CEMeR
Caminhos da Educação Matemática em Revista
2023 • Ano X • V 13 • n. 1 • p. 1 – 24
ISSN 1983-7399

ARTIGO ORIGINAL/ ORIGINAL ARTICLE

Maria Alice V. F. de Souza, Federal Institute of Espírito Santo, Brazil (1)

Arthur B. Powell, Rutgers University, USA (2)

Kyozaikenkyu: essential lesson planning in japanese lesson study

Kyozaikenkyu: essência do planejamento de aula no lesson study japonês

ABSTRACT
Lesson Study is a teacher training process practiced by Japanese educators in a cycle subdivided into stages. Although all stages are relevant, the first one concentrates the complexity of the process that is little known by non-Japanese educators - called *kyozaikenkyu*. This stage aims to promote more knowledge about some mathematical content and its teaching practice. The qualitative and interpretative research was about how eleven Brazilian teachers appropriated a *kyozaikenkyu* on the concept of fraction for students of the sixth year of regular Brazilian Elementary Education, during twelve consecutive weeks, promoted and supported by the authors of this article, who acted as participating observers in terms of research. The result revealed the teachers' apprehension of new knowledge about fractions, especially from the measurement perspective, in addition to learning how to conduct the practice of this teaching, paying attention to social aspects during classes and recognizing the role of resources and didactic-pedagogical materials.

Keywords: Japanese Lesson Study, *Kyozaikenkyu*, Teaching-training

RESUMO
Lesson Study é um processo de formação de professores praticado por educadores japoneses em um ciclo subdividido em etapas. Embora todas as etapas sejam relevantes, a primeira concentra a complexidade do processo pouco conhecido pelos educadores não japoneses - chamado kyozaikenkyu. Este estágio visa promover mais conhecimento sobre alguns conteúdos matemáticos e sua prática de ensino. A pesquisa qualitativa e interpretativa foi sobre como onze professores brasileiros se apropriaram de um kyozaikenkyu sobre o conceito de fração para alunos do sexto ano do Ensino Fundamental regular brasileiro, durante doze semanas consecutivas, promovido e apoiado pelos autores deste artigo, que atuaram como participantes observadores em termos de pesquisa. O resultado revelou a apreensão, por parte dos professores, de novos conhecimentos sobre frações, principalmente sob a ótica da mensuração, além de aprenderem a conduzir a prática desse ensino, atentando para os aspectos sociais durante as aulas e reconhecendo o papel dos recursos e materiais didático-pedagógicos .

Palavras-chave: Estudo de aula japonês, *Kyozaikenkyu*, Formação de professores.

(1) Doctor in Education-UNICAMP, Brazil. Professor of the graduate program in Science and Mathematics Education at the Federal Institute os Espírito Santo, Brazil.
(2) Ph.D. in Mathematics Education - Rutgers University, USA. Member of the Ph.D. Program in Global Urban Studies – Rutgers University, USA. Ph.D in Education and Learning Sciences - Rutgers University, USA.

Correspondência:
alicevfs@gmail.com (1)
powellab@newark.rutgers.edu (2)

Recebido em 31/03/2022
Aprovado em 31/08/2022

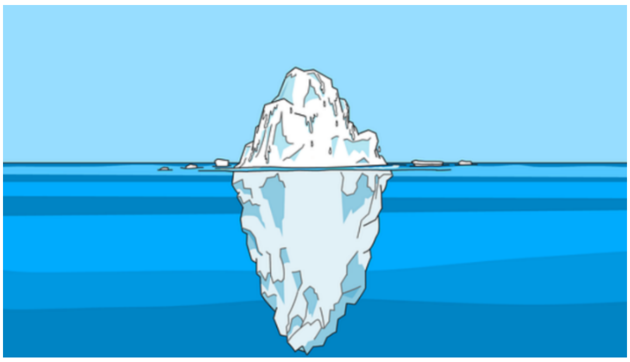
INTRODUCTION

The act of planning lessons is part of the professional daily life of teachers and is conceived by authors and researchers in Education as anticipating what is to come and designing teaching to achieve previously determined goals (Neira, 2004; Haydt, 2006; Libâneo, 1990; Sant'Anna *et al*., 1988; Isoda, 2010a; Kilpatrick, Swafford & Findell, 2001; Gaigher, Souza & Wrobel, 2017; Fernandez & Yoshida, 2004; Fujii, 2014). Although this conception is unquestionable, the content and construction of plans can vary considerably. While some plans are limited to identifying the school and teacher, objectives, contents, methodology, assessment and references, others require ingredients that contribute to the enrichment and orientation of the teacher's thinking about the content to be taught and its teaching.

That is the case of plans prepared by Japanese teachers who are part of a worldwide cycle known as Lesson Study (*jugyokenkyu*: *jugyo* = lesson*; kenkyu* = research). Lesson Study is a traditional teacher training process (*konaikenshu*: *konai* = at the school; *kenshu* = training), practiced in Japan for over 150 years, and it is developed in the three, four or more stages (Fujii, 2014, 2016; Mimura & Watanabe, 2020), depending on the stratification given by the authors for the planning of the unit or lesson (*kenkyujugyou* - lessons prepared by teachers during Lesson Study), or even for more detailing of the steps and activities undertaken by teachers[1], followed by the research lesson (*kenkyujugyo*) and critical reflection on the results of student learning (*kenkyuukyougikai*).

Although all Lesson Study stages are relevant, the construction of Japanese planning (*shidoan*) comprises an important part that concentrates the complexity of this process called *kyozaikenkyu* (*kyo* - teaching; *zai* - material; *kenkyu* - study) or, the study of the teaching material. Many projects omit the *kyozaikenkyu* part and it is crucial to "helps teachers gain knowledge and insight into mathematics and student thinking" (Takahashi & McDougal, 2016, p. 514). In this sense, when Lesson Study superficially understood just in three or four stages, it can sound like something simple, trivial and typical to the profession of any teacher. If so, it would be like just seeing part of the iceberg seen above the sea line. However, in the immersed part, we find its essence, whose ideas shape it and little is known outside Japan (Figure 1).

Figure 1 – *Kyozaikenkyu* in Lesson Study.



Source: www.pixabay.com – part.

So, what is in the immersed part of the iceberg? How does it so special in lesson planning to the point of difference from other training? How do Japanese teachers build the planning, and what are their logistics? What the Lesson Study has to offer to the point of being highlighted by researchers from around the world (Fernandez & Yoshida, 2004; Pjanic, 2014; Isoda 2011; Takahashi,

[1] In Japan, teachers and other school professionals participate in Lesson Studies, "even the school nurse and school nutritionist" (Fujii, 2016, p. 411). For ease, in this paper we will refer to these professionals simply as "teachers".

2006; Takahashi & McDougal, 2016; Fujii, 2014; Shimizu, 1999; Stigler & Hiebert, 2009; Watanabe, 2018; Takahashi, Watanabe, Yoshida & Wang-Iverson, 2005)? How can a group of non-Japanese teachers appropriate the nuances of *kyozaikenkyu*?

Given this importance, and bearing in mind that the act of teaching is a cultural activity (Hargreaves, 1996; Stigler & Hiebert, 2009; Watanabe, 2018), we developed an investigation guided by the following research question: **how do non-Japanese teachers appropriate *kyozaikenkyu* during a Lesson Study?** To study of *kyozaikenkyu*'s contributions, we will have fractions as a mathematical topic that the teachers worked on. At long last, it is essential to know the meaning of *kyozaikenkyu* and how and why it can contribute to constructing lessons carried out in the light of the Lesson Study[2]. Answers to the research question can provide significant insights for inspiring teacher training from the idiosyncrasies of Lesson Study - and particularly *kyozaikenkyu* in lessons planning - that give it a complex and broad character whose elements guide teachers in the search for better teaching. In other words, we believe that a better understanding about what Japanese Lesson Study is and, mainly, knowing how to build a potential *kyozaikenkyu*, non-Japanese teachers can be better supported to achieve high quality in instruction (Corey, Peterson, Lewis & Bukarau, 2010; Stigler & Hiebert, 2009).

RELEVANT LITERATURE

> *For Japanese teachers, lesson study is an integral part of teaching, "like the air", [...] and, as with the air, it has been hard to see what lesson study is really made of. But some aspects of Japanese lesson study have become visible as a result of flawed attempts to use it elsewhere and as a result of recent studies of jugyoukenkyuu in Japan. (Takahashi & McDougal, 2016, p. 514)*

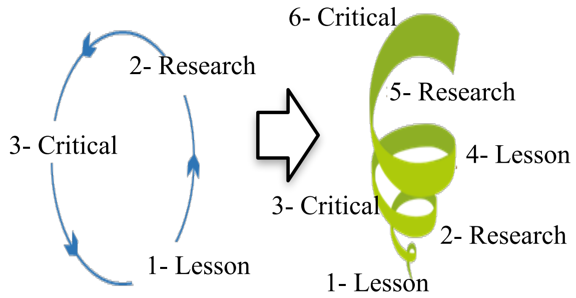
This review intends to make visible what the scientific community of Japanese educators describe and understand by *kyozaikenkyu* within lessons planning that they developed over 150 years ago in Lessons Studies. Due to the existence of the Lesson Study and the stability of its actions, this narrative review was not limited to a period, but prioritized studies and investigations by Japanese authors who have extensive experience with this type of school activity and with academic-scientific credentials in this topic (*e.g.*, Watanabe T., Fujii T., Takahashi A., Yoshida M., Isoda M., Shimizu, Y., Wang-Iverson, P.). The review, however, is supported by publications that contain keywords written in Japanese, such as *jugyokenkyu*, *kyozaikenkyu*, *neriage*, *bansho, matome*, etc. because those words have been frequent in the literature of Japanese authors.

Lesson Study is conceived as a collaborative professional development process that encourages teachers to reflect on what they know about certain school content and their teaching practice through a sequence of actions: setting goals, collaborative lesson planning, implementation and observation of the lesson plan, and reflection about students' learning. These actions may or may not be repeated if points to be improved are

[2] "*kyozaikenkyu* is the central activity in teachers' everyday practice. However, during lesson study, they [teachers] engage in *kyozaikenkyu* much more intentionally and intensively". (Watanabe, 2018, p. 5)

identified. In this case, a new cycle will start at a higher level of maturity of the professionals, such as a knowledge spiral whose actions are incorporated in the new edition of the planned, taught and critically reflected (Figure 2). Nowadays, it is not common in Japan to re-teach a research lesson (Fujii, 2014) because of collaborative actions' accumulated experience and legacy. In addition, it is helpful to keep in mind that the Lesson Study proposal is not the search for a perfect lesson plan but appropriating new knowledge for teaching and learning content that is key in the school curriculum (Takahashi, 2006; Watanabe, 2018).

Figure 2 – From the cycle to the spiral of the Lesson Study.



Source: Gaigher, 2016, our translation and adaptations.

Although the Lesson Study cycle has value for the investment result as a whole, and can look simple and obvious as part of the iceberg when seeing only the top above the sea line, it is at the base that we find the elements responsible for consistency in teacher training by shaping the content to be taught and its practice - in the lesson planning. The lesson planning in Lesson Study is concentrated at the beginning of the process. It also has the elements recommended by other authors, as we mentioned at the beginning - identification of the school and teacher, objectives, contents, methodology, assessment and references -, but the active principle for knowledge of the content and knowledge of teaching practice is located in what the Japanese call *kyozaikenkyu* or, in English, the study of instructional materials. *Kyozaikenkyu* is an initial action that can mobilize necessary knowledge about some theme or school content and its teaching practice to the point of compromising the proposal and the quality of the class to be taught.

The translation of *kyozaikenkyu* into English, or any other language, does not portray the word's full meaning in the context of a Lesson Study. Beyond what can be understood as study of instructional materials, *kyozaikenkyu* brings together actions involving investigations. The development of the study of teaching material - *kyozaikenkyu* - is a course in which teachers investigate all aspects of content and teaching materials consistent with the reasoning that students understand the topic broadly and deeply (Takahashi, Watanabe, Yoshida & Wang-Iverson, 2005). In this regard, a *kyozaikenkyu* would help broaden the understanding of any issue, depending on how it is conducted.

Globally, the material most commonly used by teachers to guide teaching is textbooks. (Beaton *et al.*, 1997; Schmidt *et al.*, 1997; Escolano & Gairín, 2005; Reys *et al.*, 2007; Alajmi, 2009; Watanabe, Lo & Son, 2017; Chingos & Whitehurst, 2012; Fan, Zhu & Miao, 2013; Robitaille & Travers, 1992; Cirillo, Drake, Herbel-Eisenmann & Hirsch, 2009; Scheffer & Powell, 2020). Indeed, they can contribute, but they are not (and should not be) the only source. If used, they should support the construction of a research lesson (Takahashi, 2006) and not as a script to teach. It is important to say that, although *kyozaikenkyu* is (or should be) a central activity in the professional daily life of teachers, in the course of a Lesson Study the dedication to the study of the material, in the Japanese sense, is much more intentional and accentuated, requiring more accurate exams on teaching and learning of curricular content and pedagogical instruments existing in educational literature (Watanabe, 2018).

The *kyozaikenkyu* should be oriented by study questions (Watanabe, Takahashi & Yoshida, 2008) based on well-defined goals and objectives for the class. For example, in lesson studies about fractions, the teachers can ask themselves: what do we want students to learn about fractions? Is a simple representation of objects sufficient? What should discussion precede or initiate the study of fractions? What knowledge do students have to follow this class? What impacts will this class have on other future mathematical and non-mathematical objects? What can different strategies be effective for learning? Textbooks are often silent on these issues, even manuals for teachers. Therefore, the teachers should investigate other sources: scientific articles, books, people experienced in the subject (knowledgeable other = *shidosha*), videos, maps, models, other teacher's lesson plans, etc., to build a research lesson and, therefore, become teachers-researchers. So, Lesson Study is more than a professional development process, but also a scientific activity (Isoda, 2010a, 2011; Watanabe, Takahashi & Yoshida, 2008; Watanabe, 2006). To Watanabe (2018, p. 10) "Japanese educators consider teaching to be research. This perspective is different from "teacher as researcher". [...] teachers are researchers of lessons".

Other Japanese educators highlight the activities developed in *kyozaikenkyu* as the essence of this part of the planning. In any case, both ideas are not contrary but complement each other:

> *Kyozaikenkyu* refers to the careful analysis of the topic in accordance with the objective(s) of the lesson. It includes analyses of the mathematical connections both among the current and previous topics (and forthcoming ones, in some cases) and within the topic. Also included are the anticipation of students' approaches to the problem and the planning of instructional activities based on the anticipated responses. (Shimizu, 1999, p. 113)

Still in the Japanese view, Watanabe, Takahashi and Yoshida (2008) tell that in the Japanese dictionary Jugyokenkyu Yougo Daijiten, written by Yokosuka in 1990, there are two views on the meaning of *kyozaikenkyu*, both focused on research activities. The first refers to the *kyozai* that are already developed for study in the different Lessons Studies. The second one is mentioned as a material that is developed for the *kyozai*. Besides, Yokosuka adds Hayashi's perception of *kyozaikenkyu* to the dictionary, which is subdivided into two stages: in the first, teachers must exhaust their understanding of a particular *kyozai* and they must know precisely what they want to teach, or what they want their students to learn from *kyozai*; in the second, teachers must develop a plan based on a particular *kyozai*. In the meantime, teachers must define, program, design, prepare, order the tasks, activities and actions that will be part of the research lesson.

Investment in *kyozaikenkyu* will likely bring many ideas and diversity of paths, but the excess of what has been investigated and(or) learned can equally compromise to achieve high quality in instruction, as much as lack or deficiency. The refinement, dosage, and essence of *kyozaikenkyu*'s emerging options must be consistent with the objectives of the unit or lesson. In this context, Watanabe, Takahashi and Yoshida (2008, p. 135) recall a famous Japanese saying, "To teach one, you have to learn ten". "Learn ten" is the main point that *kyozaikenkyu* can enrich and orient the teacher's thinking about the content to be taught and its teaching, as we said before. By "learn ten" teachers will have the opportunity to experience the traits and foundations that make up the *corpus* of *kyozaikenkyu* in the different points of view of Japanese authors, which are not contradictory or mutually exclusive, in our opinion, however when analyzed from different angles, they become complementary.

In any case, some elements are always present in *kyozaikenkyu* and offer support that those Japanese authors consider part of iceberg base. One is the use of alternative methods for solving math problems. "Lesson Study is known in the world with Problem Solving Approach" (Isoda, 2010a, p. 23). Planning in Lesson Study is supported by solving problems as a guideline for the lesson (Takahashi, 2006; Isoda & Olfos, 2009; Isoda, 2010a, 2010b). The lesson oriented intends to develop the students' ability to think and learn for themselves. In this sense, problem solving is focused on developing skill concepts much more than on the solution process itself. (Takahashi, 2006).

Due to the importance of the topic, Isoda and Olfos (2009) elaborated a checklist to help teachers implement problem solving approach in four steps: (1) Problem Posing; (2) Independent Solving; (3) Comparison and Discussion; (4) Summary. Each step has sentences that teachers should reflect on whether students achieved or not that action on a scale of 1 (not achieved) to 4 (achieved). For example, in Problem Posing step, there is a sentence that says: "The lesson sets tasks that can be solved in a variety of different ways by applying previously learned knowledge, and presenting the content to be learned". (Isoda, 2010a, p. 24) The concern here is with the teacher's repertoire to solve the problem through different strategies (as Minsky[3] says: "You don't understand anything until you learn it in more than one way"). The teacher must build the necessary stimuli for the students to appropriate these different ways of seeing and solving the same problem.

Just as teachers must pay attention to the choice of the problem and the diversity of solution strategies, other demands are present. For this, Watanabe, Takahashi and Yoshida (2008) highlight that teachers must answer some questions in the *kyozaikenkyu* process expressed in Table 1.

Table 1 – Some questions suggest to teachers answer along the *kyozaikenkyu.*

What does this idea really mean?
How does this idea relate to other ideas?
What is/are the reason(s) for teaching this idea at this particular point in the curriculum?
What ideas do students already understand that can be used as a starting point for this new idea?
Why is this particular problem useful in helping students develop this new idea?
How can students solve this problem using what they already know, and how can their solution strategies be used to develop this new idea?
What are common mistakes? Why do students make such mistakes?
How should teachers respond to those mistakes?

[3] Marvin Lee Minsky foi um cientista cognitivo estadunidense que desenvolveu estudos cognitivos no campo da inteligência artificial. https://www.brainyquote.com/quotes/marvin_minsky_106398.

What new ideas are students expected to build using this idea in the future?
What manipulatives and other materials should be provided to students? How do they influence students' learning?

Source: Watanabe, Takahashi & Yoshida, 2008, p.136.

These authors warn that these questions (Table 1) do not prevent others that are more general or specific to the school content under study from being formulated. The big idea here is to make teachers verify the breadth and depth with which they understand the object under study, and the perspective of their students. Also, teachers must keep in mind that the contents are not isolated, but connected in a curriculum as a *continuum* that makes sense to the students and, therefore, they must know the previous baggage that can support new ideas and how will be studied can support future ones in the subject of Mathematics or others. In Lesson Study, the curriculum must be studied in three main dimensions: (1) throughout the elementary school; (2) between the different units of the school year; (3) following problems of the unit under study (Isoda & Olfos, 2021).

Another attention during *kyozaikenkyu* is to students' misunderstandings. The teacher must anticipate the doubts, misunderstandings and reactions of students during the research lesson. Responding to these demands does not mean providing ready-made answers but encouraging them to think about the problem at hand. One form is what the Japanese call *hatsumon*, or "asking a key question that provokes students' thinking at a particular point in the lesson" (Shimizu, 1999, p. 109). The *hatsumon* technique goes beyond clarifying doubts or misunderstandings, it is a means of promoting understanding of the problem and connecting the various strategies that emerged from the lesson. For this, the teacher must perform *kikan-shido*.

"*Kikan-shido* means *instruction at students' desk* and includes a purposeful scanning by the teacher of the students' individual problem-solving processes" (Shimizu, 1999, p. 110). During *kikan-shido*, the teacher will collect the students' different strategies, correct or not, and this material will feed the class discussion. At this point, the teacher will assume the role of a "maestro" of the discussions that the students must carry out. This discussion will promote access to students' intellectual production, thus broadening mathematical thinking about the problem.

All knowledge production of the mathematics lesson must be arranged on a large-size blackboard (*bansho*). The blackboard works to record the synthesis of different problem-solving methods to facilitate discussions among students. In addition to the record, Yoshida (2005) highlights that the visual aid emerging from the *bansho* helps students: to remember aspects of the content; to connect different parts of the lesson; to compare, contrast and discuss the main mathematical ideas; to organize thinking and discover new ideas; to model organization of notes by students. The triggering of these variables justifies the design of the *bansho* during *kyozaikenkyu*.

After that, the final moments of the lesson are reserved for the *neriage* and the *matome*. In Japanese, *neriage* means polishing up and in the context of Lesson Study it is materialized by the students' statements about what they learned in that class. For this, teachers need to have a variety of solutions in advance to enhance discussions with students. This anticipation is performed not only with efficient methods, but also with those that generate misunderstandings. (Takahashi, 2006). Japanese educators consider *neriage* an opportunity to assess the success or failure of applying planning (Shimizu, 1999). After *neriage*, the teacher summarizes the students' mathematical knowledge production, called *matome* (summing up).

As we have seen, learning mathematical content occurs in Lesson Study through a problem (Watanabe, 2018; Fujii, 2016) or questioning that constitutes a common thread for the ideas to be developed in the lesson. This dimension led Shimizu (1999) to propose a tabular organization organized within the *kyozaikenkyu* (Table 2).

Table 2 – A common framework for lesson plans.

Steps	Main learning activities	Antecipated students' responses	Remarks on teaching
• Posing a problem • Students' individual problem solving • Whole-class discussion • Summing up (Exercise/Extension)			

Source: Shimizu, 1999, p. 113.

The Lesson Study planning is described in detail in a document called *gakushushido-an*. This fact does not mean that it is configured as a script, but as guiding the teacher's conduction in the research lesson and other teachers' observation of students' learning. Table 2 highlights at the macro level key points that should be taken into account during *kyozaikenkyu*. The problem must be proposed at the beginning of the lesson, followed by an attempt at an individual solution by the students, discussion among the students and summarized by the teacher (Watanabe, 2018). Lesson Study members should be concerned with learning activities, anticipating student responses, and observations ranging from *hatsumon* to *bansho* in each stage.

The entire *kyozaikenkyu* process is still little known to non-Japanese educators, although Lesson Study is increasingly being used by many countries. This and other unfamiliarity about lesson planning has been generating misconceptions and misunderstandings, "possibly because the effort involved is invisible to outsiders" (Fujii, 2016, p. 1), whose attention is more focused on the visible components - the research lesson and the discussion post-lesson. "In the early literature on Lesson Study, the process of *kyozaikenkyu* was obscured in lesson planning" (Watanabe, 2018, p. 5). Therefore, the focus of this work is to understand how non-Japanese educators appropriate *kyozaikenkyu* aiming to inspire and support future training worried about expanding the teacher's knowledge and the potential learning of students, based on this Japanese way of thinking and teaching lessons.

METHODS

Qualitative and interpretive research (Bogdan & Biklen, 1994) focuses on knowing in the process of a Lesson Study how non-Japanese teachers appropriate *kyozaikenkyu* when engaged in lesson planning on the concept of fraction for students in the sixth regular year of Brazilian Elementary Education.

The research took place in an optative *stricto sensu* postgraduate course in Mathematics Education at the Federal Institute of Espírito Santo – Brazil, 12 weeks long of *kyozaikenkyu* development during the planning of classes that were subsequently taught and critically reflected on student learning outcomes over four consecutive weeks.

Eleven in-service elementary teachers participated in the course - seven are graduates in Mathematics and four are graduates in Pedagogies - and two Mathematics Education researchers – authors of this article. Due to postgraduate studies, the 11 teachers worked in Brazilian Elementary Education and were not far from their professional commitments. The spontaneous engagement of teachers in the Lesson Study was motivated by the recognition of difficulties in conducting classes on fractions, thus justifying the investment in this process of professional and scientific training of the participants. The Lesson Study was promoted and supported by the authors of this work - the first author participating in person in the group, and the second one at a distance - both acting as participant observers in terms of investigation, and at the same time becoming knowledgeable other (or, in Japanese language, *shidosha*), concerning the process of teacher education. The participation and involvement of knowledgeable others in Lesson Studies is relevant to "provide guidance to a team during planning or give final comments at the end of a post-lesson discussion that help participants learn from the lesson". (Watanabe, 2018, p. 6)

The 12 face-to-face planning meetings lasted three hours each. In between these meetings, there was a need for long-distance interactions (*e.g.*, whatsapp, email, skype) with the Lesson Study participants to share materials, clarify questions, complementary debates, in short, collaboration in the development of *kyozaikenkyu*. The planning meetings - in person and at a distance - were recorded in audio, image and logbook, which constituted the raw data of the investigation, whose interpretation was carried out based on the essence of the *kyozaikenkyu* on fractions, notably in the appropriation of four main aspects: (1) of the mathematical content; (2) teaching practice (*e.g.*, planning, physical aspects, timing, *kikan-shido*); (3) attention to social particularities during classes (*e.g.*, student-student and student-teacher interaction - *neriage*, *matome*), (4) resources and materials (*e.g.*, *bansho*, *kyozai*). In each aspect, we sought to capture the teachers' understanding of the items suggested in Table 1 (Watanabe, Takahashi & Yoshida, 2008, p. 136), recording, whenever appropriate, the conceptions and understandings prior and after the development of *kyozaikenkyu*.

According to the literature presented, these four aspects should receive attention from the group of teachers during Lesson Studies planning. The stratification, however, took place for the purposes of organizing the analysis because, in the practice of *kyozaikenkyu*, those aspects are not treated in a linear way, but in a constant coming and going.

RESULTS AND ANALYSIS

Mathematical content of fractions

As a starting point, the first face-to-face meeting was guided by free debate among teachers about the difficulties in fractional classes that motivated them to participate in the Lesson Study. In general, the teachers' view was that students did not understand the basic arithmetic operations with fractions - addition, subtraction, multiplication and division. They stated that students often added numerators and denominators without considering the equivalence of fractions and the least common multiple (*e.g.*, $1/2 + 2/3 = 3/5$). Others added students' difficulties in applying fractions in mathematics problems and, for this reason, they constantly requested the teacher's help to indicate the mathematical operation to be performed to solve the problems.

The free debate of the first meeting emphasized the students' conceptions and learning. Little was said about what they knew about fractions at that time. Thus, researchers questioned teachers about the content of fractions in order to lead them to self-knowledge. The statements that emerged from there could guide the *kyozaikenkyu* on aspects to be studied for the potential of the classes. Although our intention was to capture the teachers' knowledge, the questions were sometimes focused on student learning, aiming to minimize possible constraints. The questions and the essence of the answers are presented in Table 2.

Table 3 – Questions and answers about the content of fractions.

	Questions	**Answers**
1	For your students to understand fractions, what prerequisites for mathematical ideas do you think are important for them to know?	"Natural numbers, multiplication, division, least common multiple, greatest common divisor, prime factor decomposition."
2	What do you think your students should learn about fractions?	“Representation, equivalent, comparison, addition, subtraction, multiplication and division, proper fractions, improper, apparent, simplification”.
3	What are your students' most significant difficulties or questions about fractions?	“Addition, subtraction, multiplication, division, improper fractions, equivalent, least common multiple”.
4	What is your concept of a fraction?	“A part or a piece of a whole”, “A rational number”, “Division between two integers”.
5	What representations do you offer your students to help them visualize fractions and their arithmetic operations?	“Pizza pieces, cakes”, “Chocolate bars”, “Circles”.
6	What are the differences between the properties of integers and fractions?	"The division of two integers is not always an integer." "In rationals, a division of whole numbers will always be rational." "I'm not clear about the differences."
7	How do these differences influence fractional learning?	“Students end up making some mistakes, for example: when comparing fractions with equal numerators and different denominators, they mark the largest fraction as the one with the highest denominator”. "When adding fractions, they add numerators and denominators indiscriminately." "Half the class understands 'how to do it' or 'how to solve it', and the other half doesn't even understand that."

8	How does knowledge of fractions shape more advanced content in mathematics?	"I do not know". "Knowledge of fractions is useful for teaching rational numbers in decimal representation." "I don't directly associate fractions with other contents". "It enables the understanding of irrational numbers, equations, and geometry issues, among others."
9	How can knowledge of fractions influence understanding of equations and inequalities?	"I do not know". "When the information given represents part of the integer that must be determined, especially when dealing with solutions within the set ℚ and ℝ". "For me, fractions are important because they 'appeared' in equations and inequalities".
10	A teacher, realizing his students' difficulties in operating with fractions, decides to transform them into decimals. What do you think of this decision?	"The idea of dividing the numerator by denominator is a viable option for teaching fractions". "An alternative to teaching". "I learned that way in textbooks". "It's not always a good idea because some decimals are periodic."

Source: Authors' file.

The teachers' answers (Table 2) lead us to understand that: (1) the fraction conception of the participants is supported by the part-whole and quotient perspectives; (2) the students' difficulties are concentrated in the four basic arithmetic operations, improper fractions, equivalence and least common multiple; (3) participants know little about the properties of numbers and the impacts and contributions of learning fractions on other mathematical content; (4) representations are restricted to parts of pizzas, cakes, chocolate bars and circles.

In short, the teachers did not seem to understand the actual meaning, the connections with other ideas, and why to teach fractions. In general, they understood that mastery of natural numbers, factorization, and basic arithmetic operations were sufficient to begin the study of fractions, and that the part-whole perspective was the way to get there. There was, at this time, no suggestion by the group on how to change the *status quo* of fraction classes. For this reason, researchers asked how we would power student learning. After a long silence, the researchers asked how they planned classes and the answer was unanimous: "for the school textbooks" (Brazilian public schools receive textbooks from the Ministry of Education after evaluation and approval by the Book and Material Didactic National Program).

In order to broaden the vision of the content, the researchers proposed studies in scientific articles, non-didactic books, videos and other materials that would feed the debates in the group and inspire them for the design of the classes. As an initial contribution, the researchers provided some scientific articles and books dealing with the topic (*e.g.*, Caraça, 1951; Siegler *et al.*, 2012; Siegler *et al.*, 2013; Siegler & Lortie-Forgues, 2015; Powell, 2019a, 2019b; Mack, 1993; Fujii, 2016; Harvey, 2017). Other works were sought by the teachers themselves and joined the list of studies at *kyozaikenkyu*. The professor-researcher profile seemed to be incorporated into the planning of classes, in addition to the traditional development in investigations restricted to the scope of postgraduate studies. Each teacher volunteered to study part of the material, present it and discuss it with the group.

These readings during *kyozaikenkyu* soon pointed out reasons for the school failure of this content and signaled teaching from a perspective that teachers were unaware of - the measuring perspective. This dynamic predominated in the following meetings due to the exhaustive sharing of knowledge emerging from this diversity of materials, which opened up new points of view about mathematical content, expanding and shaping their old conceptions. The main aspects discussed by teachers from the stimuli generated by the sources studied are summarized in nine main items:

1- **Impact of fractions on further studies**: the study of fractions integrates future mathematics, especially in algebra and probability (*e.g.*, $y = (3/5)\,x$; 4/7 of $84; what is the probability of guessing 5 numbers out of 60?). Research indicates that the domain of fractions can influence future school performance in Mathematics.

Main sources: Siegler *et al*. (2011); Siegler *et al*. (2012); Siegler *et al*. (2013); Hackenberg and Lee (2015).

2- **Importance of fraction comparison**: the quantity and magnitude of the digits of the numerator or denominator do not determine the magnitude of the fraction: *e.g.*, $8/9 > 123/432$, being $8 < 123$ and $9 < 432$.

Source: Powell (2019b).

3- **Properties of natural and fractional numbers**: $\mathbb{N}$ and $\mathbb{Q}$ have particular properties. For example, natural numbers have a single immediate predecessor and successor; fractional numbers, no (*e.g.*, in $\mathbb{N}$, 2 is the only immediate successor of 1; in $\mathbb{Q}$, there are infinite fractions between 1/2 and 1/3).

Main sources: Mack (1993); Siegler *et al*. (2011); Siegler *et al*. (2013); Powell (2019a, 2019b).

4- **Part-whole interpretation**: The study of fractions from the part-whole perspective suggests an instructional emphasis on two distinct numerical parts - numerator and denominator. As a result, students often get 3/8 as a result of 1/3 + 2/5, and do not understand the fraction 5/3, as it is impossible to get 5 parts of an object divided into 4 equal parts. In fractional multiplication, students often confuse when and why the denominators are kept leading to errors of the type $2/5 \times 3/5 = 6/5$. Research indicates that the part-whole perspective is not sufficient for understanding the rational number system, mathematically and psychologically.

Main sources: Fujii (2016); Lamon (2001); Siegler *et al*. (2013); Powell (2019a).

5- **Interpretation by measuring**: Unlike the part-whole perspective - which emphasizes counting objects -, the understanding of fractions by measuring works on the idea of comparing continuous quantities by their magnitudes. Children develop an intuitive proportional understanding in the context of continuous quantities before they can manipulate discrete sets or numerical proportions. Therefore, the initial study of fractions by magnitudes is indicated. This scientific argument had repercussions on the motivation for the knowledge of improper fractions, equivalence of fractions, least common multiple and arithmetic operations for continuous quantities.

Main sources: Caraça (1951); Jeong, Levine and Huttenlocher (2007); Siegler *et al*. (2013).

6- **Part-whole *vs* measuring**: The initial study from the measuring perspective brings epistemological advantages over the part-whole perspective by understanding the relationship between numerator and denominator and not as separate parts.

Main sources: Siegler *et al*. (2011); Siegler *et al*. (2013).

7- **Conceptual knowledge *vs* procedural knowledge**: conceptual knowledge includes an understanding of properties, magnitudes, principles and notations; procedural knowledge relies on algorithms.

Source: Siegler *et al*. (2013).

8- **Symbolic knowledge *vs* non-symbolic knowledge**: Even though students have an innate ability to process non-symbolic reasons (*e.g.*, five points), in many countries, students are unable to take advantage of this ability to build robust knowledge of fractions for being stimulated primarily by symbolic knowledge (*e.g.*, number 5).

Source: Powell (2019b).

9- **The unit and the number line**: The notion of unit plays an important role in constructing conceptions about fractions. The lack of knowledge of the unit as a reference for contexts of fractional numbers can lead to mistakes such as proposing the comparison between 1/2 and 1/3 of the same object with different sizes. In this case, 1/3 can be greater than 1/2 depending on the unit considered in each fraction. Positioning on the numerical line helps to understand the magnitudes: 1/2 can be positioned at different places on the line depending on the defined unit.

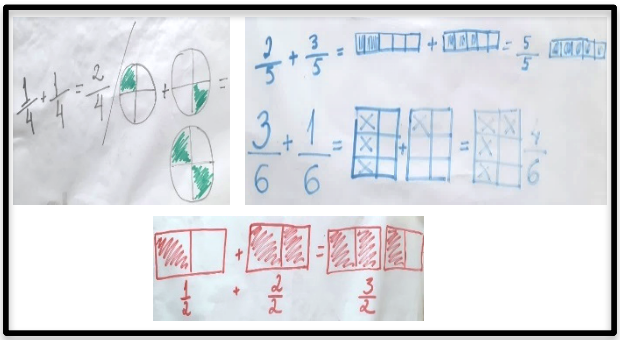
Source: Fujii (2016).

In addition to understanding the origin of the misunderstandings presented by the students in the classes of those teachers, the studies seem to have indicated a mathematical core that constituted a starting point for the design of lessons in the Lesson Study: Understanding the magnitudes of fractions leads to relationships that the count does not offer. Therefore, the idea of continuous quantities should precede the teaching of discrete quantities. This conclusion strongly impacted the group of teachers who asked themselves: "Why don't books [Brazilian textbooks] follow this line of thought [of the magnitudes of fractions]?".

Fraction teaching practice

After the free debate and questioning of the first meeting (Table 2), the researchers asked the teachers to show each other how they used to teach some topic of fractions to Elementary Education students. For this, the researchers provided flipchart sheets of paper and marker pens. Figure 3 shows some productions that teachers repeated with few variations in their essences. These productions reinforce the teachers' conception of fraction in the part--whole perspective.

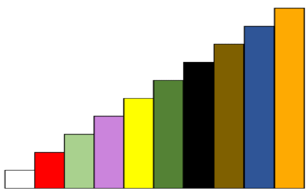
Figure 3 – Teachers' presentation on teaching fractions.



Source: Authors' file.

After the presentations, the teachers started the *kyozaikenkyu* whose studies of mathematical content concurrently offered clues on how to teach fractions through the idea of continuous quantities (*e.g.*, Caraça, 1951; Powell, 2019a, 2019b). These clues and other investigations about this teaching (*e.g.*, Powell, 2018; Watanabe, 2018) guided the group to construct the concept by measuring. Teachers opted for the use of Cuisenaire rods and acquired kits sold in the Brazilian market (Figure 4).

Figure 4 – Cuisenaire rods.



Source: Authors' file.

Teachers discovered from the small manual that accompanied the kits that the rods could help with other Mathematics content (*e.g.*, proportions, symbols, measures, ten, hundred, addition, subtraction, multiplication and division, column chart, fraction, etc.). Amidst these discoveries, a discussion provoked by the researchers led the group to reflect: what is the role of Cuisenaire rods for the proposal of teaching fractions? After a long silence, some teachers highlighted that the material could lead students to understand fractions, others highlighted the motivating role. Certainly, manipulative materials can function as allies to teaching, but more critical than the materiality of the chosen objects are the intended mathematical objectives that underlie them.

The discussion culminated in some main conclusions: (1) concrete materials are of secondary importance, as mere manipulation does not guarantee learning improvement, and (2) materials should support mathematical ideas that challenge students. Mathematical knowledge should be at the forefront and not materials. At that moment, one of the teachers asked the others: "What [mathematical] ideas are going to be taught?". One of the researchers replied about the need for a pedagogical proposal, without which the class could be essentially empty and, therefore, motivation and teaching could be innocuous.

The researchers proposed the study of a pedagogical approach developed by Powell (2018) called 4A-Instructional Model. This proposal offers a possibility to develop the concept of fraction from the perspective of measuring using Cuisenaire rods. It is composed of four actions that are not necessarily linear, but progressive in terms of the maturation of ideas: Actual Actions, Virtual Actions, Actions Written, Actions Formalized (Powell, 2018).

The group wanted to know the 4A-Instructional Model. The researchers, therefore, developed the four actions with the group of teachers, leading them to experience the situations that the students would experience. In possession of the rods and under the guidance of the researchers, the group developed Actual Actions' ideas: freely manipulate the rods to build length relationships (Figure 5a) and introduce a code to compare the magnitudes of the rods (Figure 5b).

Figure 5 – (a) discovery of rod length relationships; (b) code for comparing magnitudes.

(a)

(b)

white	red	light green	purple	yellow	dark green	ebony	tan	blue	orange
w	r	g	p	y	d	e	t	b	o

Source: Authors' file.

Then, in Virtual Actions, the teachers mentally manipulated the images aiming to internalize their relationships. The presence of mistakes, insecurities or embarrassments for this activity led them to return to Actual Actions, whenever necessary. Also, in the Virtual Actions stage, teachers were encouraged to create magnitude relations between the rods without viewing them. Mastering the first two actions enabled them to proceed to Actions Written when mathematical equations and inequalities were introduced, representing the magnitudes of the relationships built in the previous steps (Figure 6). The last step consisted of the symbolic formalization of mathematical ideas and procedures with the magnitudes of the rods.

Figure 6 – Examples of equations and inequalities that represent relationships between magnitudes.

$p < t$ $\qquad$ $d = 3r$ $\qquad$ $y > w + g$

Source: Authors' file.

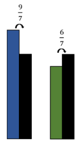
The practical experience provided by 4A-Instructional Model using Cuisenaire rods in *kyozaikenkyu* led teachers to reflect on how to develop the concept of fraction. To this end, the group exhaustively planned six lessons of 100 minutes each. The lesson plan was guided by the steps of the 4A-Instructional Model, aiming at the development of topics that would develop the initial concept of fractions, highlighting: (1) passage from proper to improper fractions (Figure 7a); (2) comparisons of proper, improper and mixed fractions with the same denominator (Figure 7b); (3) equivalent fractions (Figure 7c); (4) least common multiple (Figure 7d); (5) comparisons of fractions with different denominators (Figure 7e); (6) comparison of fractions with equal numerators, as a particular case of the previous comparison (Figure 7f).

Figure 7 – Fractions teaching.

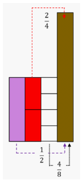
(a) Example of representation of the transition from proper fractions to improper ones.

Magnitude of the colored rods in relation to the white rod representing 1 unit of measure.	Fractions 1/8, 2/8, 5/8, 8/8, 9/8 e 13/8.

(b) Comparison example of improper and proper fraction with same denominator: 9/7 > 6/7.



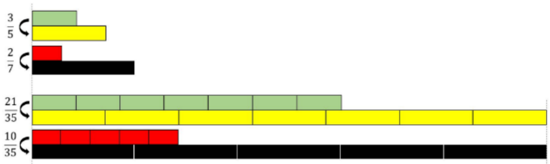
(c) Example of equivalent fractions 1/2, 2/4 e 4/8.



(d) Example of least common multiple between 5 e 7.

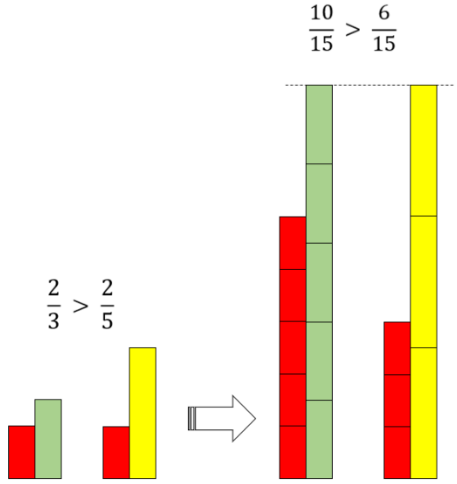
Step	Least common multiple between 5 e 7	Quantity of rods
1		1 yellow 1 ebony
2		2 yellows 1 ebony
3		2 yellows 2 ebonies
4		3 yellows 2 ebonies
5		3 yellows 3 ebonies
6		4 yellows 3 ebonies
7		5 yellows 3 ebonies
8		5 yellows 4 ebonies
9		6 yellows 4 ebonies
10		6 yellows 5 ebonies
11		7 yellows 5 ebonies

(e) Example of comparing fractions with different denominators (3/5 > 2/7; 21/35 >



10/35) - (Source: Amaral, 2020, p. 37).

(f) Example of comparing fractions with equal numerators (2/3 > 2/5; 10/15 > 6/15).



Source: Authors' file.

The group considered that the six classes would be the basis for continuing the study of arithmetic operations with fractions. Furthermore, the ideas developed there would leave a legacy for algebraic thinking by constructing of equations and inequalities without formally mentioning them.

Pay attention to social particularities during classes

Social aspects that contribute to the effectiveness and potential of learning must be considered in *kyozaikenkyu*. In this context, the group should plan: (1) student-student and teacher-student interactions, (2) the physical disposition of students and teacher-observers (teachers participating in the Lesson Study who would not lead the class) in the classroom , (3) how to prevent external interruptions that could impair the reasoning flow during classes, (4) the moment of *kikan-shido* to collect and understand the knowledge production of students in the various proposed activities that would feed the *neriage* and the *matome*, between other arrangements.

The group planned interactions primarily through *hatsumon*. For example, during the Virtual Actions, students were scheduled in pairs to question one another (with eyes closed) about the magnitudes of the rods. In general, activities would be carried out collaboratively and gathered in groups. The students' productions would be presented and discussed by them to the whole class upon the teacher's (teacher-executor) invitation. The physical arrangement of the desks should facilitate this interaction and displacement of the teacher-executor to access the development of activities (*kikan-shido*). As the space in the students' classroom did not meet the purposes of the Lesson Study, the group decided to do them in the school library.

The teacher-observers would not interfere with the course of the classes, but should record the actions in images and audio. To facilitate silent communication between teacher-observers, a group on whatsapp was created to share images and information in real-time.

External interruptions were expected at the public school where classes would be held. The teachers planned to ask the school principal not to stop classes, except in dire need. During this discussion, teachers remembered that students often ask to drink water and go to the bathroom frequently. To avoid this transit, teachers agreed to ask students to do those things before starting class, when absence would then be allowed in exceptional cases.

Resources and materials

Although Japan is a highly technological country, classes at Elementary School are not usually conducted by projectors, computers, tablets, cell phones, televisions or any other resources of this type. For them, the first years of school must have paper, pen and mind[4]. Contrary to these resources, the whiteboard gains special attention for recording the entire production of knowledge in the class (*bansho*). In other words, students must have access to everything they have built during the class, so the whiteboards are never erased and are often prominent in Japan.

With that in mind, teachers pondered the fact that there were no whiteboards in the school library and so planned to use flipchart sheets, just as they showed each other how they would teach fractions at the beginning of Lesson Study meetings. The sheets would be taped to the library walls.

The restriction on the use of technological resources in Japan does not prevent the use of manipulative materials that help with the objectives of the class. The teachers understood that the Cuisenaire rods would support a greater purpose: the construction of the concept of fraction.

DISCUSSION

Based on the protocols of the participants during the *kyozaikenkyu*, it is possible to affirm that there was an apprehension of new knowledge about fractions. The conception of fractions from the perspective of measurement brought possibilities for solving the learning difficulties of students reported in the initial stage of the Lesson Study and which even justified continuing education. By the way,

[4] In a teaching and research meeting, the first author asked Prof. Isoda why Japan does not use digital technologies in the classroom, being a highly technological country. He replied that in elementary school, students only need paper, pen and mind.

overcoming the limitations in the study of students' fractions does not seem to find support in teachers' praxis, considering the conceptual limitations presented before *kyozaikenkyu*.

In this regard, the community of researchers in Mathematics Education and Cognitive Psychology has been denouncing gaps in the understanding of fractions by teachers and students, and advising on ways to remove them (*e.g.*, Powell, 2019a, 2019c; Zhou, Peverly & Xin, 2006; Maher & Yankelewitz, 2017; Toluk-Uçar, 2009; Putra, 2016; Hoof *et al*., 2018; Huang, Huang & Lai, 2021; Fujii, 2006; Lin *et al*., 2013; Siegler & Lortie-Forgues, 2015; Siegler *et al*., 2012; Utley & Reeder, 2011; Van Steenbrugge *et al*., 2014; Hackenberg & Lee, 2015; Bobos & Sierpinska, 2017; Alqahtani *et al.*, 2022). Access to other sources than the textbook provided clarification of flaws in their conceptions, making it potential for a new look at the content of fractions and how to teach it. The high amount of scientific material available led the participants to share the acquired knowledge in synchronous moments. They would hardly have been able to study the same volume of materials if they were alone. This fact led the participants to recognize the advantages of collaborative work and the high investment of time and effort.

Teachers were unanimous in noting the lack of knowledge about the study of fractions from the perspective of measurement (Caraça, 1951; Powell, 2019a, 2019b, 2019c). Using Cuisenaire rods, teachers understood the advantages of developing the concept of fractions to be initiated by the perspective of measurement and continuous quantities. This fact is reinforced in the investigative results of some investigators, especially in Alqahtani *et al.* (2022) and Amaral (2021).

The work with the rods provided, in a natural way, the construction of equations and inequalities when comparing the magnitudes of the fractions (see Figure 6). This fact broadened the view on the impacts of the study of fractions on future curricular contents. Some teachers mentioned the desire to use this knowledge to introduce new content (equations and inequalities) based on what they learned from the rods. However, considering what it means to study the curriculum in Lesson Study, the view of fractions of teachers seems to be still incipient. There seems to have been no apprehension of the content throughout the elementary education curriculum as a whole, not even inside and outside the unit of study in that school year.

By way of teaching, the group considered the number of classes that would be necessary for dedication to the development of the fraction concept based on the three properties studied and using Cuisenaire rods. At first, there was insecurity in the feasibility of development with students due to the small number of classes for this purpose in schools. The completion of the planning made them see that *kyozaikenkyu's* time was much greater than the amount of classes for concept development with the students. In practice, they understood the weight of the Japanese quote "To teach one, you have to learn ten".

The discovery of the 4A-Instructional Model as a pedagogical approach to what they wanted to teach led to the expansion of the perception of "problem". Problem-based classes, in the Japanese conception, do not necessarily mean to be word problems (see, Souza & Guimarães, 2015) or of the "calculate" type, as is usual in Brazilian mathematics textbooks. The "problem" we are talking about is some situation that is new and that challenges students to think and discover for themselves what they should learn. This seems to be the true spirit of the "problem" from a Lesson Study perspective. "Problem Solving Approach [...] is a well-known theory of teaching for developing children who learn mathematics by/for themselves in Japan" (Isoda, 2010a, p. 1). In this sense, the four Actions of the 4A-Instructional Model promoted challenging situations for the students and, therefore, there was a common thread for the lessons planned by the group.

As for social particularities, *kikan-shido* was not new because it is already practiced by many teachers when organizing their students to develop group activities. The originality came with the collection of student production data to feed discussions with the class after the end of activities - individual or not - launched by the teacher. Furthermore, the planning of the *neriage* and the *matome* generated some discomfort in the group. Classes in Brazil are interrupted for external reasons or due to the closing of time. In Japan, classes are like stories: with a beginning, a middle and an end. The synthesis of what was learned by the students (*neriage*) and the summary offered by the teacher (*matome*) is what ends the class. Classes in Japan are not interrupted by external factors. They are respected and almost understood as "sacred" (Stigler & Hiebert, 1999). This was a sensitive factor for teachers because they have no control over these interruptions. In this case, the group decided to ask the direction of the school for surveillance to not allow interruptions. As stated by Watanabe (2018, p. 1) "when a practice that is successful in one culture is imported into another, however, there are usually some obstacles and challenges, and this has been the case with a Japanese lesson study".

The detailed record (*gakushushido-an*) of the teacher's actions, the students' reactions and doubts, and the questions (*hatsumon*) led the teachers to reflect on the need for training in the face of unusual and unpredictable situations during class. Some situations happened during *kyozaikenkyu* in the context of using Cuisenaire rods and acted as warning and forecast registered in *gakushushido-an*. For example, when students are asked to show with rods all the possibilities of representation of the fraction 1/2, how could the teacher act if someone fails to show all the possibilities? The teachers' first suggestions were for broad questions that did not help the student to reason (*e.g.*, did you make all the representations possible? The group reflected and chose to have the student try them one by one: the white rod is 1/2 of what other rod? the red rod is 1/2 of what other rod? and so on). Broad questions lead students to a mental search trying to know what the teacher wants them to answer (see, Souza; Guimarães, 2015). Unlike questions aimed at the objective of the moment (*e.g.*, in problem situations: "what does the problem want to know?", "what is the problem data?" etc.).

In short, *kyozaikenkyu*: (1) added novelties to what was known about the content and practice of teaching fractions; (2) provided clarification of doubts and solutions for students' learning difficulties; (3) it provided opportunities for reflections on how to act throughout the class, how to conduct students' reasoning and how to orchestrate student-student interactions; (4) expanded the view on the range of options for studying content that exceeds textbooks; (5) broadened the reading of the impacts of the study of fractions for(in) future mathematics (*e.g.*, equations, inequalities, functions, etc.).

IMPLICATIONS

The results of this investigation suggest that it is possible to carry out the Lesson Study process outside Japan and to reverse flaws that persist in teacher education that generate negative capillarities for students' learning. Simply looking at fractions as part-whole is not enough to teach improper fractions, equivalence, and arithmetic operations with fractions. This limitation can be overcome with the measurement perspective, as directed by the specific scientific community.

FUTURE RESEARCH

This investigation leads to other issues that deserve to be scientifically developed. How do teachers plan lessons in their schools after the Lesson Study experience? What facilities, constraints and challenges do they find in schools that teach classes? Would the fact that participants in this edition of the Lesson Study are graduate students differ in performance during *kyozaikenkyu*? What was the result of applying the planning with the sixth graders? Well, the answer to that last question already exists and can be found in "Frações à Moda Antiga" (written in Portuguese language: bit.ly/3d9kLpA), but that's another story.

ACKNOWLEDGMENT

The research reported here is part of a larger project developed with financial support from Global Rutgers (New Jersey - USA).

REFERENCES

Alajmi, A. H. (2009). Addressing computational estimation in the Kuwaiti curriculum: teachers' views. *Mathematics Teacher Education*, *12*(4), 263-283.

Alqahtani, M. M., Powell, A. B., Webster, V., & Tirnovan, D. (2022). How a Measuring Perspective Influences Pre-service Teachers' Reasoning about Fractions with Discrete and Continuous Models. *International Electronic Journal of Elementary Education*, 14(3), 441-458.

Amaral, C. A. N. (2021). *Conceito de fração pela perspectiva de medição: uma abordagem baseada no 4A-Instructional Model utilizando as barras de Cuisenaire e conduzida por um lesson study*. [Dissertation, Federal Institute of Espírito Santo].

Beaton, A., Mullis, I., Martin, M., Gonzalez, E., Kelly, D., & Smith, T. (1997). *Mathematics achievement in the middle school years: IEA's Third International Mathematics and Science Study (TIMSS)*. Center for the Study of Testing, Evaluation, and Educational Policy.

Bobos, G., & Sierpinska, A. (2017). Measurement approach to teaching fractions: a design experiment in a pre-service course for elementary teachers. *International Journal for Mathematics Teaching and Learning*, *18*(2), 203-239.

Bogdan, R., & Biklen, S. *Investigação qualitativa em educação*. Porto Editora.

Caraça, B. J. (1951). *Conceitos fundamentais da Matemática*. Tipografia Matemática.

Chingos, M. M., & Whitehurst, G. J. (2012). *Choosing blindly: instructional materials, teacher effectiveness, and the Common Core*. Brookings Institution.

Cirillo, M., Drake, C., Herbel-Eisenmann, B., & Hirsch, C. (2009). Curriculum vision and coherence: adapting curriculum to focus on authentic mathematics. *Mathematics Teacher*, *103*(1), 70-75.

Corey, D. L., Peterson, B. E., Lewis, B. M., & Bukarau, J. (2010). Are there any places that students use their heads? Principles of high-quality Japanese mathematics instruction. *Journal for Research in Mathematics Education*, *41*, 438-478.

Escolano, R. E., & Gairín, J. M. (2005). Modelos de medida para la enseñanza del número racional en Educación Primaria. *Revista Iberoamericana de Educación Matemática*, *1*, 17-35.

Fan, L., Zhu, Y., & Miao, Z. (2013). Textbook research in mathematics education: development status and directions. *ZDM Mathematics Education*, *45*, 633–646.

Fernandez, C., & Yoshida, M. (2004). *Lesson Study: a Japanese approach to improving mathematics teaching and learning*. Autores Associados.

Fujii, T. (2016). Designing and adapting tasks in lesson planning: a critical process of lesson study. *ZDM Mathematics Education*, *48*, 411-423.

Fujii, T. (2014). Implementing Japanese lesson study in foreign countries: misconceptions revieled. *Mathematics Teacher Education and Development*, *16*(1), 65-83.

Gaigher, V. R. (2016). *Formação do professor de matemática em aulas de resolução de problemas a partir de ações colaborativas e reflexivas*. [Dissertation, Federal Institute of Espírito Santo].

Gaigher, V. R., Souza, M. A. V. F., & Wrobel, J. S. (2017). Planejamentos colaborativos e reflexivos de aulas baseadas em resolução de problemas verbais de matemática. *Vidya*, *37*(1), 51-73.

Hackenberg, A. J., & Lee, M. Y. (2015). Relationships between students' fractional knowledge and equation writing. *Journal for Research in Mathematics Education*, 46(2), 196–243.

Hargreaves, A. (1996). *Changing teachers, changing times: teachers' work and culture in the postmodern age*. Cassell.

Harvey, R. (2017). Stretching student teachers' understanding of fractions. *Mathematics Education Research Journal*, *24*(4), 493-511.

Haydt, R. C. (2006). *Curso de didática geral*. Ática.

Hoof, J. V., Degrande, T., Ceulemans, E., Verschaffel, L., & Dooren, W. V. (2018). Towards a mathematically more correct understanding of rational numbers: a longigutinal study with upper elementary school learners. *Learning and Individual Differences*, *61*, 99-108.

Huang, X., Huang, R., & Lai, M. Y. (2021). Exploring teacher learning process in Chinese lesson study: a case of representing fractions on a number line. *International Journal for Lesson & Learning Studies*. DOI: 10.1108/ijlls-03-2021-0026.

Isoda, M. (2010a). Lesson Study: Japanese problem solving approaches. In *Conference on reaplicating exemplary practices in Mathematics Education.* Koh Samui.

Isoda, M. (2010b). Lesson Study: problem solving approaches in Mathematics Education as a Japanese experience. *Procedia Social and Behavioral Sciences*, *8*, 17-27.

Isoda, M. (2011). Problem solving approaches in Mathematics Education as a product of Japanese Lesson Study. *Journal of Science and Mathematics Education in Southeast Asia*, *34*(1), 2-25.

Isoda, M., & Olfos, R. (2009). *El Enfoque de Resolución de Problemas: en la enseñanza de la matemática a partir del Estudio de Classes*. Ediciones Universitarias de Valparaíso.

Isoda, M., & Olfos, R. (2021). *Teaching multiplication with Lesson Study: Japanese and Ibero-American theories for international Mathematics Education*. Springer. DOI: 10.1007/978-3-030-28561-6.

Jeong, Y., Levine, S. C., & Huttenlocher, J. (2007). The development of proportional reasoning: effect of continuous versus discrete quantities. *Journal of Cognition and Development*, *8*(2), 237–256.

Kilpatrick, J., Swafford, J., & Findell, B. (Eds.). (2001). *Adding it up: helping children learn mathematics*. National Academy Press.

Lamon, S. J. (2001). Presenting and representing: from fractions to rational numbers. In Cuoco, A., & Curcio, F. (Eds.). *The roles of representations in school mathematics - 2001 yearbook*. (pp. 146-168). National Council of Teachers of Mathematics.

Libâneo, J. C. (1994). *Didática*. Cortez.

Lin, C-Y, Becker, J., Byun, M-R., Yang, D-C, & Huang, T-W. (2013). Preservice teachers' conceptual and procedural knowledge of fraction operations: a comparative study of the United States and Taiwan. *School Science and Mathematics*, *113*(1), 41-51.

Mack, N. K. (1993). Learning rational numbers with understanding: the case of informal knowledge. In Carpenter, T. P., & Fennema, E. (Eds.). *Studies in mathematical thinking and learning - Rational numbers: an integration of research* (pp. 85-105). Lawrence Erlbaum.

Maher, C. A., & Yankelewitz, D. (2017). *Children's reasoning while building fraction ideas*. Sense Publishers.

Mimura, K., & Watanabe, Y. (2021). *Developing an evaluation framework in lesson study on active learning methods.*

Neira, M. G. (2004). *Por dentro da sala de aula: conversando sobre a prática*. Phorte Editora.

Pjanic, K. (2014). The origins and products of Japanese Lesson Study. *Teaching Innovations*, *27*(3), 83-93.

Powell, A. B. (2018). Melhorando a epistemologia de números fracionários: uma ontologia baseada na história e neurociência. *Revista de Matemática, Ensino e Cultura* (REMATEC), *13*(29), 78-93.

Powell, A. B. (2019a). How does a fraction get its name? *Revista Brasileira de Educação em Ciências e Educação Matemática*, *3*(3), 700-713.

Powell, A. B. (2019b). Aprimorando o conhecimento dos estudantes sobre a magnitude da fração: um estudo preliminar com alunos nos anos iniciais. *International Journal for Research in Mathematics Education*, *9*(2), 50-68.

Powell, A. B. (2019c). Measuring perspective of fraction knowledge: integrating historical and neurocognitive findings. *Revista Sergipana de Matemática e Educação Matemática*, *4*(1), 1-19.

Putra, Z. H. (2016). Evaluation of elementary teachers' knowledge on fraction multiplication using anthropological theory of the didactic. In *13th International Congress on Mathematical Education*. Hamburg.

Reys, B. J., Chval, K., Dingman, S., McNaught, M., Regis, T. P., & Togashi, J. (2007). Grade-Level Learning Expectations: a new challenge for elementary mathematics teachers. *Teaching Children Mathematics*, *14*(1), 6-11.

Robitaille, D. F., & Travers, K. J. (1992). International studies of achievement in mathematics. In Grouws, D. A. (Ed.). *Handbook of mathematics teaching and learning* (pp. 687-709). Macmillan Publishing Company.

Sant'Anna, F. M. *et al*. (1998). *Planejamento de ensino e avaliação*. Sagra.

Scheffer, N. F., & Powell, A. B. (2020). Frações na educação básica: o que revelam as pesquisas publicadas no Brasil de 2013 a 2019. *Revista Paranaense de Educação Matemática*, *9*(20), 8-37.

Schmidt, W., McKnight, C., & Raizen, S. (1997). *A splintered vision: an*

investigation of U.S. science and mathematics education*. Kluwer.
Shimizu, Y. (1999). Aspects of mathematics teacher education in Japan: focusing on teachers' roles. *Journal of Mathematics Teacher Education*, *2*, 107-116.
Siegler, R. S., Thompson, C. A., & Schneider, M. (2011). An integrated theory of whole number and fractions development. *Cognitive Psychology*, *62*, 273-296.
Siegler, R. S. *et al*. (2012). Early Predictors of High School Mathematics Achievement. *Psychological Science*, *23*(7), 691-697.
Siegler, R. S., Fazio, L. K., Bailey, D. H., & Zhou, X. (2013). Fractions: the new frontier for theories of numerical development. *Trends in Cognitive Sciences*, *17*(1).
Siegler, R. S., & Lortie-Forgues, H. (2015). Conceptual knowledge of fraction arithmetic. *Journal of Educational Psychology*, *107*(3), 909-918.
Souza, M. A. V. F.; Guimarães, H. M. (2015). A formulação de problemas verbais de matemática: porquê e como. *Quadrante*, *24*(2), 135-162.
Stigler, J., & Hiebert, J. (1999). *The teaching gap: best ideas from the world's teachers for improving education in the classroom*. Free Press.

Takahashi, A. C. (2006). Characteristics of Japanese mathematics lessons. *Tsukuba Journal of Educational Study in Mathematics*, *25*, 37-44.
Takahashi, A., & McDougal, T. (2016). Collaborative lesson research: maximizing the impact of lesson study. *ZDM Mathematics Education*, *48*, 513-526.
Takahashi, A.; Watanabe, T.; Yoshida, M.; Wang-Iverson, P. (2005). Improving content and pedagogical knowledge through Kyozaikenkyu. In: Wang-Iverson, P, & Yoshida, M. (Eds.). *Building our understanding of lesson study* (pp. 101-110). Research for Better Schools.
Toluk-Uçar, Z. (2009). Developing pre-service teachers understanding of fractions through problem posing. *Teaching and Teacher Education*, *25*(1), 166-175.
Utley, J., & Reeder, S. (2011). Prospective elementary teachers' development of fraction number sense. In Wiest, L. R., & Lambert, T. (Eds). 33rd Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education... , Reno. *Proceedings...* University of Nevada.
Van Steenbrugge, H., Lesage, E., Valcke, M., & Desoete, A. (2014). Preservice elementary school teachers' knowledge of fractions: a mirror of students' knowledge? *Journal of Curriculum Studies*, *46*(1), 138-161. DOI: 10.1080/00220272.2013.839003.
Yoshida, M. (2005). Using lesson study to develop effective blackboard practice. In Wang-Iverson, P., & Yoshida, M. (Eds.). *Building our understanding of lesson study* (pp. 93-100). Research for Better Schools.
Watanabe, T. (2018). Japanese lesson study in the United States. *Journal of the International Society for Design and Development in Education*, *3*(11).
Watanabe, T. (2006). The teaching and learning of fractions: a Japanese perspective. *Teaching Children Mathematics*, *12*(7), 368-374.
Watanabe, T., Lo, J.-J., & Son, J.-W. (2017). Intended treatment of fractions and fractions operations in mathematics curricula from Japan, Korea and Taiwan. In Son, J.-W. (Ed.). *What Matters? Research Trends in International Comparative Studies in Mathematics Education* (pp. 33-62). Springer International Publishing.
Watanabe, T., Takahashi, A., Yoshida, M. (2008). Kyozaikenkyu: a critical step for conducting effective lesson study

and beyond. In Arbaugh, F., & Taylor, P. M. (Eds.). *Inquiry into Mathematics Teacher Education* (pp. 131-142). Association of Mathematics Teacher Educators (AMTE).

Zhou, Z., Peverly, S. T., & Xin, T. (2006). Knowing and teaching fractions: a cross-cultural study of American and Chinese mathematics teachers. *Contemporary Educational Psychology*, *31*(4), 438-457.