CEMeR
Caminhos da Educação Matemática em Revista
2023 • Ano X • v 13 • n. 1 p. • 25-46
ISSN 1983-7399

ARTIGO ORIGINAL/ ORIGINAL ARTICLE

Measuring fences and sharing pizzas: current advances in nonsymbolic fraction interventions

Medindo cercas e compartilhando pizzas: avanços atuais nas intervenções de fração não simbólica

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Recebido em 31/03/2022
Aprovado em 31/08/2022

ABSTRACT
Students finalizing elementary school should successfully solve fraction arithmetic and comparison problems. However, less than 30% of eighth-grade students in the United States have accomplished these educational milestones. Fraction knowledge is vital to learning more complex math and has major implications on health and employability. Thus, it is crucial to develop educational methods to enhance fraction understanding. In contrast to these struggles with symbolic fractions, young children have surprisingly strong nonsymbolic proportional reasoning, a capacity positively related to fraction ability in older children and adults. These findings have led to educational interventions that leveraged nonsymbolic skills, primarily via numberlines, to enhance symbolic fraction understanding. In a systematic review of recent studies, we identified 22 studies (from 19 articles), which we grouped into three categories: classroom-based interventions, multiple-session training, and single-session training studies. These studies provide an optimistic picture of the malleability of students' fractions skills. Placing nonsymbolic and symbolic representations of fractions on numberlines enhanced the fraction skills of low- and typically-achieving students, resulting in small-to-large effect sizes. These results suggest that fostering nonsymbolic skills may be necessary to address the persistent bottleneck fractions pose in mathematical knowledge.

Keyword: educational interventions, fractions, nonsymbolic representations, proportional reasoning

RESUMO
Os alunos que concluem o ensino fundamental devem resolver com sucesso problemas de aritmética de frações e de comparação. No entanto, menos de 30% dos alunos de 13 anos nos EUA coseguem fazer isso. O conhecimento de frações é vital para aprender matemática mais complexa e tem grandes implicações na saúde e na empregabilidade. Por isso, é crucial desenvolver métodos educacionais para melhorar a compreensão das frações. Em contraste com esses desafios com frações simbólicas, as crianças pequenas têm um raciocínio proporcional não simbólico surpreendentemente forte, uma capacidade positiva relacionada à habilidade de compreender fração em crianças mais velhas e adultos. Essas descobertas levaram a intervenções educacionais que alavancaram habilidades não simbólicas, principalmente por meio de linhas numéricas, para melhorar a compreensão da fração simbólica. Em uma revisão sistemática de estudos recentes, identificamos 22 estudos (de 19 artigos), que agrupamos em três categorias: intervenções baseadas em sala de aula, treinamento em sessões múltiplas e estudos de treinamento em sessão única. Esses estudos fornecem uma imagem otimista da maleabilidade das habilidades de frações dos alunos. A adoção de representações não simbólicas e simbólicas de frações em linhas numéricas melhorou as habilidades de fração de alunos de desempenho baixo e típico, resultando em efeitos pequenos a grandes. Esses resultados sugerem que a promoção de habilidades não simbólicas pode é importante para melhorar a compreensão das frações.

Palavras-chave: intervenções educativas, frações, representações não simbólicas, raciocínio proporcional

INTRODUCTION

Fractions pose a particular challenge to students as they require *expanding* and *refining* their understanding of numbers (Rosenberg-Lee, 2021). For mathematically correct comprehension, students must expand their numerical magnitude representations to include whole numbers, fractions, and rational numbers more generally, as well as refining rules that formerly applied to whole numbers but not to fractions (e.g., larger numerals, larger numerical quantity). In fact, only a minority of students show a correct understanding of fractions, even after several years of fraction instructions. For example, less than 30% of eighth-graders in the United States correctly solved a fraction arithmetical problem (Carpenter et al., 1980; Lortie-Forgues et al., 2015) even though they should have established this skill by the end of elementary school (Common Core State Standards Initiative, 2020). Thus, developing educational methods to enhance students' fraction understanding has become critical.

In contrast to the challenges that fractions pose, nonsymbolic representations of proportions are more easily understood by students (Abreu-Mendoza et al., 2020; Hurst & Cordes, 2018; Jeong et al., 2007). In fact, children as young as four years of age can work successfully with proportions presented nonsymbolically (Hurst & Cordes, 2018). Consequently, recent interventions have attempted leveraging these nonsymbolic skills to enhance fraction understanding. In this paper, we first present the growing number of studies showing that children have an understanding of nonsymbolic proportions before formal fraction instruction, then an overview of the correlational studies that posit this understanding as a building block for fraction knowledge. These correlational studies provide a foundation for intervention and training studies, which offer a powerful methodological tool to establish causal links between proposed foundational abilities and more complex outcome skills (Rosenberg-Lee, 2018), in this case, nonsymbolic and symbolic proportional abilities, respectively. Next, we conduct a systematic search for recent intervention studies that have used nonsymbolic representations of proportions to promote students' fraction knowledge. We synthesize their findings and consider the role that different nonsymbolic representations (e.g., area models and numberlines) played. Finally, we discuss how these intervention and training studies can provide insights on the nature of the relationship between nonsymbolic and symbolic representations of fractions and future directions for new intervention studies.

NONSYMBOLIC SKILLS AS A BUILDING BLOCK FOR FRACTION KNOWLEDGE

Do children understand proportional magnitudes before any formal instruction about fractions?

Students' struggles with fraction knowledge could be due to a lack of understanding of how the new symbolic framework that fractions introduce work (e.g., learning the relation between numerators and denominators) but also to a lack of understanding of their meaning (i.e., proportional thinking). While students may have an implicit conceptual understanding of whole-number quantities from birth (Carey, 2009; Feigenson et al., 2004; Piazza, 2010), it has generally been assumed that understanding proportional quantities requires extensive formal instruction as these types of quantities are a recent cultural artifact and children may lack any implicit knowledge of them (Hartnett & Gelman, 1998). However, over the last two decades, a growing body of evidence suggests that children understand nonsymbolic proportional magnitudes from a very early age. From six months of age, infants can discriminate changes in the proportion of a number of objects (McCrink & Wynn, 2007): babies are surprised when, after repeatedly watching collections of dots showing a constant proportion of yellow and blue dots (4:1 ratio), the collections switch to show a different proportion (2:1). Before elementary school, children can successfully perform additions and subtractions of nonsymbolic proportions (Mix et al., 1999), compare two continuous nonsymbolic proportions (Hurst & Cordes, 2018), identify objects of differing sizes but the same proportions (Boyer et al., 2008), and solve analogies using proportional information, for example, recognizing that half a circle is equal to half a rectangle as a quarter of a circle is equal to a quarter of a rectangle (Goswami, 1989).

Young children's remarkable performance on proportional reasoning task (*e.g.*, proportions of liquid) contrasts sharply with their performance when they have to work with discrete proportions (*e.g.*, segmented bars) (Abreu-Mendoza et al., 2020; Begolli et al., 2020; Boyer et al., 2008; Hurst & Cordes, 2018; Jeong et al., 2007). In particular, children struggle when reasoning about discrete proportions that contradict whole-number knowledge (*e.g.*, 3/4 vs. 4/9). These type of proportions may highlight part-whole relationships, which in turn, leads to invalid counting strategies (Plummer et al., 2017). Only after age nine, can children perform above chance levels in this type of proportional reasoning problems (Begolli et al., 2020; Jeong et al., 2007). Together, these studies suggest that different nonsymbolic formats tap into different aspects of proportional reasoning skills, which may bear on their relationship with fraction knowledge.

Is there a relationship between symbolic and nonsymbolic skills?

These studies provide considerable evidence that children have conceptual knowledge about proportions from a very young age. However, most of these studies did not examine whether children's nonsymbolic proportional skills relate to their knowledge of symbolic fractions. In fact, there is scant evidence showing such a relationship. Two studies have found that children's skills at solving nonsymbolic proportional reasoning problems are related to a general assessment of fraction knowledge (Begolli et al., 2020; Möhring et al., 2016). In particular, elementary-school children who were better at either placing a proportion represented as a bicolored bar on a numberline (Möhring et al., 2016) or matching objects of different sizes but the same proportions (Begolli et al., 2020) had higher scores in fraction knowledge assessments, including fraction arithmetic and fraction comparison problems, respectively.

A stronger test of the relationship between nonsymbolic and symbolic proportional skills is to examine whether proportional magnitude understanding is related across formats. That is, whether understanding of nonsymbolic proportional quantities is related to that of equivalent symbolic fraction quantities. One of the most common methods to examine magnitude understanding, regardless of the quantity type (whole or rational) or format (symbolic or nonsymbolic), is quantity comparison tasks, as they afford measurement of the precision of these representations (Halberda et al., 2008; Schneider et al., 2017). Among the first studies to tackle this question, Matthews et al. (2016) assessed college students' ability to compare nonsymbolic proportional quantities, using different representations (bars and circles), and the ability to compare fractions. Students with more accurate nonsymbolic proportional comparison skills also had more accurate fraction comparison skills. However, more recently, in an effort to replicate these findings, Park and Matthews (2021) found further evidence for the relationship between nonsymbolic proportional skills and general measures of fractions knowledge and math achievement but failed to find supporting evidence for the relationship between symbolic and nonsymbolic proportional comparison skills.

In summary, prior research suggests that students' difficulty with learning fractions may not stem from a lack of understanding of proportional magnitudes, as children can work easily with nonsymbolic proportions. However, different formats of nonsymbolic proportions may be more effective in promoting fraction knowledge (e.g., continuous formats), while others may introduce some challenges (e.g., discrete formats). Interestingly, Begolli et al. (2020) found that there were stronger correlations between discretized and symbolic performance than continuous and symbolic performance. More importantly, the evidence for the relationship between symbolic and nonsymbolic proportional skills is still mixed, and there is a need to establish the nature of the possible causal link between these constructs.

Casual links can be examined through training and intervention studies (Rosenberg-Lee, 2018). In particular, these studies test whether intervening on nonsymbolic skills leads to improvements on symbolic skills, suggesting a causal relation between them. Thus, to further our understanding of the nature of this relationship, we reviewed studies that focused on enhancing students' nonsymbolic proportional abilities or used nonsymbolic representations as a scaffold for fraction understanding.

METHODOLOGY

To conduct the systematic review, in November 2021, we searched for relevant literature via the Scopus database (see Figure 1), using the following combinations of search terms: 1) fraction magnitude AND training; 2) fraction magnitude AND intervention, 3) nonsymbolic AND proportions AND fractions AND intervention; 4) and nonsymbolic AND proportions AND fractions AND training, resulting in 297 documents. Notably, although we did not constrain our search to a publication year range, the combination of these terms resulted in a limited range of publication years, from 2016 to 2022[1]. This restricted range may reflect the more recent use of the search terms (e.g., nonsymbolic) in the fraction intervention literature. Next, we narrowed the list of articles by removing duplicates (*n* = 83) and eliminating titles that were unrelated to fraction intervention (*n* = 189).

From the 25 remaining documents, we first confirmed that these documents were published in English-language peer-reviewed journals then checked for the following inclusion criteria: 1) Conducted randomized controlled or quasi-experimental intervention, 2) included outcomes of

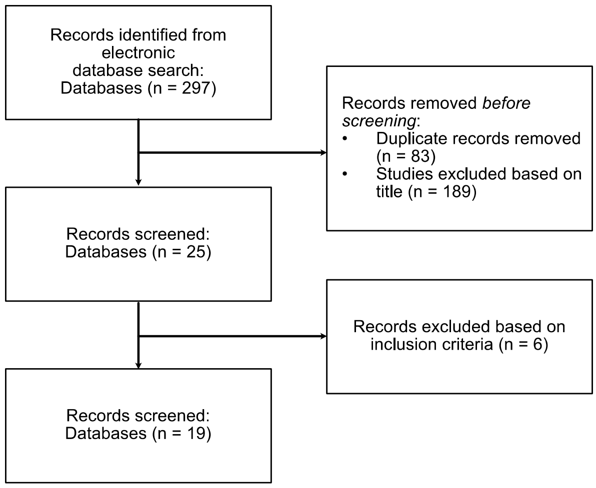
[1]One paper, Hurst et al. (2022), that was in press in November 2021 when we conducted our search has since been published.

nonsymbolic or symbolic proportional reasoning as the dependent variable, and 3) included an intervention with fraction teaching or an experimental training condition involving nonsymbolic representations of fractions as one of their primary independent variables. Six studies were removed in this step: one study focused on the effects on teachers implementing the intervention instead of the students who completed it, four studies only reported descriptive statistics, and one did not use nonsymbolic representations systematically.

Finally, we classified the remaining 19 papers, which presented 22 studies, into the following three categories: classroom-based interventions (*n* = 7), multiple-session training studies (*n* = 5), and single-session training studies (*n* = 7; Figure 2). The first category involved interventions that were implemented in the classroom by trained teachers who, in most cases, went through professional development training. The last two categories comprised training sessions mainly conducted by researchers and differed in the number sessions.

Across papers, outcome measures were typically nonsymbolic and symbolic assessments of proportional skills. Nonsymbolic measures involved either comparing two nonsymbolic representations of proportions or placing the proportional magnitudes represented nonsymbolically in a numberline. Symbolic measures were more varied, including placing fractions on a numberline, comparing pairs of fractions, performing fraction arithmetic, solving word problems that involve fractions, ordering fractions, and understanding the density of rational numbers. Standardized measures were also employed, such as the Test for Understanding of Fractions (TUF, Instructional Research Group, 2014, 2015) or items from the National Assessment of Educational Progress (NAEP). For this review, when studies had multiple outcome measures, we report those that were consistent across studies of the same category. Papers also collected a host of domain-specific measures, such as standardized assessments of general math ability (e.g., Wide Range Achievement Test–4, Wilkinson & Robertson, 2006), whole-number magnitude estimation, geometry knowledge, and nonsymbolic proportional skills and domain-general ones, such as working memory, inhibitory control, receptive vocabulary, class attentive behavior, and nonverbal reasoning, which is beyond the scope of this review.

Figure 1 – PRISMA flow diagram illustrating search procedures.



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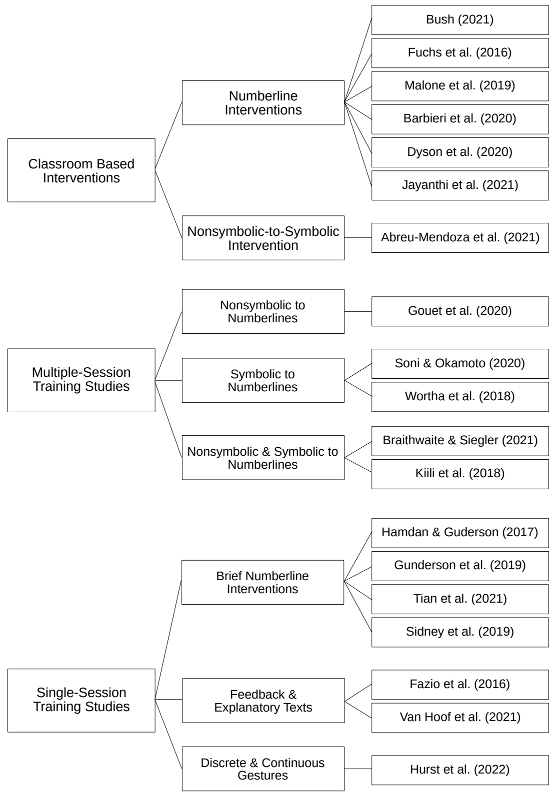
Papers reported effect sizes using a variety of statistics. Most studies used standard effect sizes, Cohen's *d,* Hedges' *g,* and partial η^2. For Cohen's *d* and Hedges' *g*, values of 0.2, 0.5, and 0.8 can be considered small, medium, and large effects, respectively (Cohen, 1988). Partial η^2 of .01, .06, and .14 are considered small, medium, and large effects, respectively. Papers also reported non-standard measures of effect sizes. Fuchs et al. (2016) used the "difference between adjusted posttest means divided by the pooled SD of the unadjusted posttest scores" (p. 502); Malone et al. (2019) used the "absolute mean difference (controlling for pretest) divided by the residual variance within the control arm only" (p. 7). Gouet et al. (2020) used a difference between the pre- and post-training scores, divided by the standard deviation of the pre-training scores across all participants. These last measures of effect sizes are presented in the text as ES and can be roughly interpreted the same way as Cohen's *d*.

RESULTS

Classroom-based interventions

Papers in this category involved long-duration interventions in students' classrooms, that were typically conducted by trained teachers. First, we focused on the six papers that employed numberlines as their primary instructional tool (Numberline Interventions, Figure 2), in which participants were students who had already received some fraction instruction (Barbieri et al., 2020; Bush, 2021; Dyson et al., 2020; Fuchs et al., 2016; Jayanthi et al., 2021; Malone et al., 2019). Then, we described one paper that centered on developing an understanding of proportional magnitudes using nonsymbolic representations of children who have not received fraction instruction, (Nonsymbolic-to-Symbolic Intervention, Figure 2) (Abreu-Mendoza et al., 2021).

Figure 2 –. Diagram illustrating the three study categories.



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Fractions are formally introduced in the third grade in the U.S., although part-whole relationships are presented as early as first grade (Common Core State Standards Initiative, 2020). Therefore, as all the papers in the first set involved U.S. fourth-to-sixth-grade students, these participants had likely already received some formal fraction instruction by the time of the intervention. Another common feature across these papers was that most studies focused on low-achieving students. Specifically, participants in Fuchs et al. (2016) and Malone et al. (2019) were fourth-grade students with general math difficulties (below the 35^{th} percentile in a standardized math assessment). Participants in Barbieri et al. (2020) and Dyson et al. (2020) were sixth graders who had low fraction skills in a screening assessment comprised of fraction items from the NAEP. Similarly, participants in Jayanthi et al. (2021) were fifth graders with low fraction skills (15th to 37th percentile) according Test for Understanding of Fractions. Only in Bush (2021), participants were not preselected based on their math abilities, as the intervention was delivered to the whole classroom.

The average total intervention time was 18.97 hours (*SD* = 6.91, range 10 to 30 hours) split over an average of 30 sessions (*SD* = 14.46, range 10 to 52 sessions), with each session's average duration 40.83 minutes (*SD* = 10.21, range 35 to 60 minutes). Notably, sample sizes across studies were larger than those from the other two categories: The average sample size was 73 students per condition (SD = 47.07, range 23 to 210 students).

The general goal of these six papers was to improve children's understanding of fraction magnitudes using nonsymbolic representations as visual aids. However, interventions varied in the number of topics that they covered. While some papers focused on specific skills (e.g., fraction arithmetic), others covered a comprehensive number of fraction-related topics.

From this set of six interventions, the shortest and most focused paper was Bush (2021). In this study, fourth-to-fifth graders completed a computerized intervention, *Woot Math Adaptive Learning* (Montero et al., 2018), which comprised solving fraction equivalence problems along with addition and subtraction problems. In each session, students first saw a short video instructing them how to use the digital manipulatives (*i.e.,* numberlines, pie charts, fraction bars) to solve the corresponding task; then, students completed tasks related to the topic of the video. Students in the control group continued with their usual activities. Notably, this study used a crossover design; that is, after students in the experimental group completed the intervention, those in the control group completed the intervention while participants in the experimental group returned to their usual activities. The pre-post assessment consisted of a general fraction knowledge assessment comprised of fraction numberline, equivalence, comparison, order, and arithmetic problems. Students in the intervention group had higher gains than students in the control condition in the general fraction knowledge assessment (Cohen's d = 0.39); however, after the control group completed the intervention, there were no differences between the two groups, suggesting the intervention enabled the control group to catch up to the experimental group.

In addition to being longer than Bush (2021), the next five intervention papers had several common features. All involved working with paper-and-pen activities in small groups. They also had two pre-post assessment tasks in common, a numberline task and a fraction arithmetic task. In Fuchs et al. (2016), fourth graders in the intervention groups completed the *Fraction Face-Off!* (Fuchs & Schumacher, 2011) intervention and either explanation (EXP condition) or word problem activities (WP condition). Students in the control group continued with the district math curricula. The *Fraction Face-Off!* intervention focused on building comparison, order, and equivalence abilities for proper, improper and mixed fractions, from a measurement perspective and using numberlines. However, in the first sessions of the intervention, students were also familiarized with the part-whole interpretation. Other nonsymbolic aids used by the intervention were fraction circles and fraction bars. Students in the EXP condition were taught a four-step strategy to compare fraction magnitudes, while students in the WP condition learned to recognize different types of fraction word problems and the appropriate strategies to solve them. Students in the EXP (ES = 0.63) and the WP

(ES = 0.71) outperformed children in the control group in a 0-2 numberline task. In a follow-up study, Malone et al. (2019) contrasted a *Fraction Face-Off*-based intervention that integrated fraction and decimal instruction (DM group) with another intervention group that focused on fraction applications using word problems (FAPP group) and a regular classroom activities control group. Here, fourth-grade students in both groups, FAPP (ES = 1.07) and DM (ES = 1.10), had a higher performance than control-group students in 0-2 numberline task.

In two papers by Jordan and colleagues (Barbieri et al., 2020; Dyson et al., 2020), six-grade students with low fraction skills completed an intervention that focused on proper and improper fractions using numberlines (Figure 3A) but also used fraction bars. During this intervention, participants performed the following activities: counting by unit fractions, dividing linear, area and set models into smaller pieces, solving fraction arithmetic problems (addition and subtraction), and equating mixed number to improper fractions. As fraction understanding requires strong multiplication skills, children also practiced multiplication problems to promote their arithmetic fluency. In contrast, children in the control groups worked with fraction-related software. Notably, in both papers, the intervention group outperformed their corresponding control group in a fraction numberline task (Barbieri et al., Hedges' g = 0.85; Dyson et al., Hedges' g = 0.90), which comprised 0-1 and 0-2 trials.

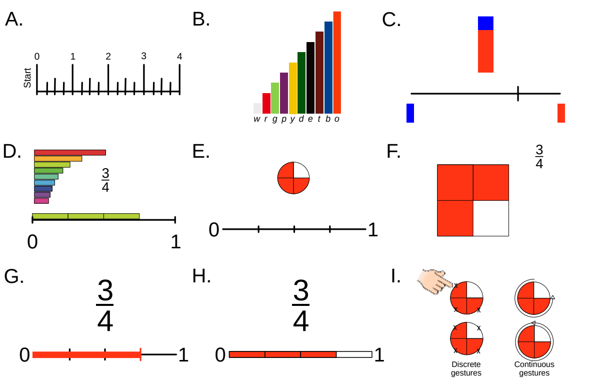
In Jayanthi et al. (2021), fifth-grade students in the intervention group completed lessons that covered topics from Grade 4 and 5 from the Common Core Standards (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). In particular, the intervention focused on fraction equivalence, ordering, and arithmetic problems with fractions with same denominators (Grade 4) and then moved on to fraction arithmetic of unlike denominators, involving mixed, improper and proper fractions. Students were encouraged to reason about fraction magnitudes throughout the intervention by highlighting the relationship between numerator and denominator and using benchmarks to place fraction magnitudes on a numberline. In contrast, students from the control group completed non-related fraction activities. Children in the intervention group had greater fraction estimation skills than children from the control group in a 0-1 numberline task (Hedges' g = 1.08) and a 0-2 numberline task (Hedges' g = 0.80). In summary, this set of five studies demonstrates that fraction interventions that include numberline activities lead to greater fraction estimation ability compared to performance in the respective control groups.

Fraction arithmetic problems were also used to assess the effectiveness of these intervention programs. However, tasks varied on the type of arithmetic operations included. Fuchs et al. (2016) and Malone et al. (2019) used only addition and subtraction problems; Dyson et al. (2020) and Barbieri et al. (2020) also included multiplication problems; and Jayanthi et al. (2021) included the four operations. Moreover, not all interventions included fraction arithmetic instructions, specifically Malone and Fuchs focused on fraction concepts, making fraction arithmetic a far transfer task. However, regardless of all these differences, across studies students from the intervention groups showed greater fraction arithmetic skills than those of the control groups. Fuchs reported that children from both experimental conditions outperformed children in the control condition (EXP: ES = 1.98; WP: ES = 2.08). Remarkably, by the end of the intervention, students in the experimental conditions performed equally well as typically achieving children. Similarly, students from the two intervention conditions in Malone et al. showed greater arithmetic skills than children in the control group (FAPP: ES = 3.14; DM: ES = 2.63). Dyson found greater performance from the intervention group (Hedges' g = 0.48); while Barbieri et al only found a small, not significant improvement (Hedges' g = 0.17). Finally, students in Jayanthi et al. showed greater arithmetic skills than children in the control group by the end of the intervention (Hedges' g = 1.07). Overall, these studies suggest that fraction numberline interventions are powerful educational tools to improve students' understanding of fraction magnitudes; however, the variability in the fraction arithmetic tasks across studies complicate assessing the effectiveness of these interventions.

In contrast to these numberline studies, the Nonsymbolic-to-Symbolic study (Figure 2) (Abreu-Mendoza et al., 2021) used physical manipulatives (Figure 3B). The intervention first focused on developing children's understanding of proportional magnitudes using nonsymbolic continuous representations, Cuisenaire rods, to then link it with symbolic representations. In particular, children first worked with Cuisenaire rods to represent and compare proportions; then, they progressively replaced these rods with written representations of proportions. Participants of this study were second-graders who had yet to receive fraction instructions, participating in an afterschool program where the intervention. To measure children's improvements in nonsymbolic proportional magnitude understanding, children completed the Spinners task (Jeong et al., 2007), in which they indicated which of two doughnut-shaped, bicolored figures had the proportionally larger magnitude. Children in the intervention group improved their ability to compare nonsymbolic continuous proportions; however, they also decreased their performance when comparing segmented proportions. For children in the control group, who did not participate in the afterschool program, there were no changes in performance on either task. These results suggest that the same intervention can have differential effects on nonsymbolic proportional skills depending on their format.

Figure 3 – Schematic renderings of training stimuli and tasks. A) In Barbieri et al. (2020), numberlines were introduced in the context of a racing game. B) In Abreu-Mendoza et al. (2021), students represented proportions using Cuisenaire then used symbols (letters) to represent them. C) In Gouet et al. (2020), children placed the proportional magnitude of the red area (3/4) in a number line that went from completely blue to completely red. D) In Braithwaite and Sigler (2021), students used fraction bars to represent the values of individual fractions or the product of fraction additions. E) In Kiili et al.'s (2018) training study, students placed nonsymbolic magnitudes and fractions and decimals (not shown) in numberlines. In Gunderson et al.'s papers (Gunderson et al., 2019; Hamdan & Gunderson, 2017; Tian et al., 2021), students worked with different nonsymbolic representations: F) square are model, G) traditional thin numberline, and H) hybrid numberline. I) In Hurst et al. (2022), children saw discrete or continuous gestures representing proportional quantities.

Training stimuli and tasks



Source: Authors' file.

MULTIPLE-SESSION TRAINING STUDIES

Training programs in this category followed a similar format: all comprised computer-based training in which students practiced with numberlines to improve their symbolic fraction knowledge on a variety of tasks. Four studies involved participants who were elementary school students (Braithwaite & Siegler, 2021; Gouet et al., 2020; Kiili et al., 2018; Soni & Okamoto, 2020); while one study examined the functional brain changes associated to fraction training in young adults (Wortha et al., 2020). The average length of the total training time was 84 minutes (*SD* = 42.78, 40 to 150 minutes), split over 2 to 5 sessions.

Participants of the elementary school studies were fourth-to-sixth-grade students from Chile (Gouet et al., 2020), Finland (Kiili et al., 2018), and the United States (Braithwaite & Siegler, 2021; Soni & Okamoto, 2020). By the time of the training, all participants should have received some formal fraction instruction, as this type of number is introduced in third or fourth grade in these countries. However, Soni and Okamoto (2020) noted that some fourth-grade participants of their study had not yet received formal fraction instruction. The average samples size was 30 students (*SD* = 12.92, range = 16 to 53 students) in each group.

Across these papers, training sessions consisted of placing quantities in a numberline, particularly, placing a) nonsymbolic (Gouet et al., 2020), b) symbolic (Soni & Okamoto, 2020) or c) both nonsymbolic and symbolic stimuli (Braithwaite & Siegler, 2021; Kiili et al., 2018). In Gouet et al. (2020), fourth-grade students practiced placing nonsymbolic proportional quantities (Nonsymbolic to Numberlines, Figure 2), either presented as bicolored bars (Study 1, Figure 3C) or blue and yellow sets of dots (Study 2), in a numberline. By contrast, children in the control group practiced placing nonsymbolic whole-number quantities in numberlines. In Study 1, children in the training condition showed a pre-to-post improvement in their ability to place nonsymbolic quantities in a numberline (ES = 0.75) that was greater than that of the control group. Study 2 replicated the findings from Study 1: children again showed a pre-post improvement in their nonsymbolic proportional skills (ES = 0.55). Remarkably, children in both studies also showed improvements in their symbolic skills. Particularly, children in Study 1 (ES = 0.29) and Study 2 (ES = 0.60) showed improvements in symbolic fraction assessment comprising ordering and equivalence problems as well as fraction arithmetic.

Soni & Okamoto (2020) focused only on symbolic fractions (Symbolic to Numberlines, Figure 2). Children in the training group placed fraction magnitudes in two-dimensional numberlines (*i.e.,* unsegmented, thin rectangles with 0 and 1 as endpoints) using either computer-based or paper-based materials, while the control group continued with their usual class activities. Importantly, the training provided progressive clues if children were unable to provide a correct response; for example, the unsegmented numberline was changed to a segmented one to facilitate a response. Regardless of the format of the training, children who worked with numberlines showed an improvement in their scores on a symbolic fraction assessment comprising fraction equivalence and comparisons problems ($\eta^2 = .33$).

Other papers of this category have combined nonsymbolic and symbolic quantities in the same study (Nonsymbolic & Symbolic to Numberlines, Figure 2). In Braithwaite and Siegler (2021), fourth-to-sixth-grade students in the main training condition used fraction bars of different sizes to indicate the position in a numberline of the outcome of symbolic fraction addition problems. Children in the comparison groups either used fraction bars to indicate the individual value of an individual symbolic fraction (Study 1, Figure 3D) or the outcome of whole-number addition (Study 2). In Study 1, children from both conditions showed similar improvements in their numberline estimations of individual fractions (Cohen's $d = 0.62$), but children who practiced with fraction addition problems showed a greater improvement in their numberline estimation of unequal-denominator sums (Cohen's $d = 1.52$ vs. Cohen's $d = 0.85$). Similarly, in Study 2, children who practiced with fraction sums showed greater improvements than those who practiced with whole-number sums (Cohen's $d = 1.72$ vs Cohen's $d = 0.85$).

In the training condition of Kiili et al. (2018), fourth graders played a game that consisted of two tasks: in one, they had to move a character to a specific location in a numberline which corresponded to the value of a decimal, a fraction, or a nonsymbolic proportional quantity, depicted as a pie chart (Figure 3E); the other task consisted of ordering the values of both decimals and fractions. Students in the control group attended regular math sessions. As part of the pre-post assessment, students from both groups completed a rational number test comprising four types of problems: estimation, comparison, order, and density problems with rational numbers. Children in the experimental group showed greater gains than those of the control group in the estimation (partial $\eta^2 = 0.15$) and ordering problems (partial $\eta^2 = 0.09$), and only marginal improvement in the density ones (partial $\eta^2 = 0.04$). No significant differences were found for the comparison task.

To date, there is only one paper that has investigated the results of fraction intervention at the behavioral and brain level. In Wortha et al. (2020), young adults completed an adaptation of the training program reported in Kiili et al. (2018), which focused only on fractions instead of fractions, decimals, and nonsymbolic proportions. Thus, this study is an example of the Symbolic to Numberlines category (Figure 2). Participants completed four pre-post, inside-scanner comparison tasks: numberline vs. fraction, numberline vs. numberline, fraction vs. fraction task, and a non-numerical control tasks. The behavioral results showed that adults became more precise in comparing fractions and faster when comparing numberlines vs. fractions and numberlines vs. numberlines after the five-day intervention. Remarkably, there were no brain-activation changes for the numberline vs. numberline and numberline vs. fraction task. Only the fraction vs. fraction comparison task showed pre-to-post brain changes. Brain activity increased in a set of frontoparietal areas implicated in math cognition, including the bilateral intra-parietal sulcus, the supramarginal gyrus, and the inferior and middle frontal gyrus. Across all the studies in this category, placing nonsymbolic and symbolic proportional magnitudes in numberlines successfully improved fraction skills.

SINGLE-SESSION TRAINING STUDIES

Papers in this category comprised brief (~15 min), single-session trainings conducted by researchers. We identified three groups of papers. The first group included four papers that contrasted the effects of different nonsymbolic representations (*e.g.*, numberlines vs. area models) on symbolic fraction knowledge of children at initial stages of fraction instructions, second-to-fifth-graders (Gunderson et al., 2019; Hamdan & Gunderson, 2017; Sidney et al., 2019; Tian et al., 2021). The second group comprised two papers that used numberlines in their training conditions, but an additional feature of the training was the main focus of the paper (e.g., feedback vs. no feedback) (Fazio, Kennedy, et al., 2016; Van Hoof et al., 2021). The last group comprised one paper that targeted young children's (five-to-seven-year-olds) nonsymbolic proportional abilities using continuous and discrete gestures (Hurst et al., 2022).

Across papers, the average sample size was 35.90 students (*SD* = 6.54, range = 25 to 45) for each group, and the length of the total training time ranged from 10 to 15 minutes. Moreover, training instructions from most studies were given by the experimenter, except for Van Hoof et al. (2021), which were computerized.

The first group comprised four papers examining the effects of varying different features of numberlines on children's fraction knowledge (Brief Numberline Interventions, Figure 2). Two features of the numberlines provide major advantages over area models: 1) their unidimensionality allows for placing both whole number and rational numbers in the same nonsymbolic representation, and 2) they leverage left-to-right directionality to convey fraction magnitudes. In a first study, Hamdan and Gunderson (2017) investigated whether second and third graders would show greater improvements in their fraction knowledge after receiving a numberline intervention than after receiving either an area model intervention or completing crossword puzzles. Importantly, to prevent children from counting the hatch marks of the numberline instead of the spaces between them, researchers used a thin rectangle as the numberline (hereafter hybrid numberline, Figure 3H). For their area model, they used circles. Children who placed fraction magnitudes on hybrid numberlines were more precise in their estimates of fraction magnitudes on numberlines and comparing pairs of fractions (transfer task) than children in the other two conditions (area model: Cohen's *d* = 0.84; crossword puzzles: Cohen's *d* = 0.54). These findings show that numberlines are more effective than area models in teaching fractions but leave as an open question which is the crucial feature of numberlines to convey fraction magnitudes. Thus, in a second study, Gunderson et al. (2019) compared the effectiveness of three numberline representations (the hybrid numberline, a traditional thin numberline (Figure 3G), and a square numberline (Figure 3F), which removed the unidimensionality feature but kept the left-to-right directionality) and a square area model. After the training, second and third graders who received the traditional and hybrid numberline interventions were better at the numberline task than those who completed the square numberline and area model training conditions (traditional vs. square area: Cohen's *d* = 0.91; hybrid vs. square area: Cohen's *d* = 1.09). However, children who completed the hybrid numberline showed greater transfer effects in the fraction comparison task than children in the square conditions (hybrid vs. square area: Cohen's *d* = 0.84; hybrid vs. square number line: Cohen's *d* = 0.75), suggesting that unidimensionality is a critical feature for fraction learning, but hybrid best support transfer effects.

In a third paper, Tian et al. (2021) examined whether numberlines are effective in teaching the magnitudes of improper fractions to fourth and sixth graders. Students completed one of three training conditions: a hybrid numberline, a square area model, or a non-numerical control (crossword puzzles). Children in the numberline condition did not perform better than those in the other conditions in the numberline tasks (partial $\eta^2 < .01$) and either of the two transfer magnitude comparison tasks (magnitude comparison task: partial $\eta^2 < .02$; comparison to one task: partial $\eta^2 < .02$). Notably, children in the area model training showed better performance in an area model task but no transfer effects. In the last study of this group, Sidney et al. (2019) compared the fifth-to-sixth-grade students' ability to solve fraction division problems with one of three different nonsymbolic aids (numberline, rectangular area model, or circle area model) or no aids. Students who were given numberlines as nonsymbolic aid had higher performance on the fraction divisions than those in the other three conditions. However, there were no transfer effects when students were asked to generate stories to represent fraction division problems or solve fraction division story problems.

The two papers of the second group used numberlines during their interventions but they were not the main experimental focus (Feedback & Explanatory Texts, Figure 2). Fazio et al. (2016) conducted two studies. In Study 1, fourth-to-fifth-grade students first received a 3-min instruction on how to place fractions on a numberline and then completed a computerized training which involved placing fractions in a numberline which provided performance feedback. Results from Study 1 showed that children improved their fraction estimations abilities (Cohen's *d* = 1.10), as well as their fraction comparison skills (Cohen's *d* = 0.48) and recall of fraction magnitudes from fraction problems (Cohen's *d* = 0.52), transfer tasks. To confirm that students' learning gains were due to the instruction and the feedback, Study 2 included a control condition where students placed fraction magnitudes on a numberline but did not receive instruction or feedback. Consistent with Study 1, children who received the fraction numberline instruction and feedback improved in their numberline estimations (Cohen's *d* = 0.86) and marginally improved their fraction comparison skills (Cohen's *d* = 0.35); in contrast, children in the no feedback group did not show these improvements. Interestingly, recall of fraction magnitudes improved in both groups (intervention group: Cohen's *d* = 0.49; control group: Cohen's *d* = 0.36). Together, these studies suggest that practicing placing numbers on a numberline and feedback contribute to children's fraction magnitude understanding.

Van Hoof et al. (2021) examined the effects of expositional text explaining why 3/5 is larger than 3/7 and a refutation text which advised students of the misconception of using 7 > 5 as a rule to choose the larger fraction. Both texts included numberlines to further demonstrate that 3/5 is larger than 3/7. The authors hypothesized that the refutation text will lead to greater learning gains as this text addresses a common students' misconception about fraction. However, in contrast to the researchers' hypotheses, students who received the expositional text had a (marginally) better performance in an immediate posttest and better one in a delayed posttest six weeks after than students in the refutation text.

In the last group (Discrete & Continuous Gestures, Figure 2), Hurst et al. (2022) aimed to ameliorate children's difficulty with working with nonsymbolic discrete proportions by using different gestures that either highlighted the nonsymbolic proportions' segments or highlighted the magnitude as a whole (Figure 3I). More specifically, children were introduced to a character that only liked shapes with just the right amount of color and just the right amount with no color. Then, children were introduced to the nonsymbolic proportions, presented as pie charts, but with different gestures depending on the condition. In the discrete gesture condition, children saw the experimenter pointing to the colored parts (numerator) one by one; then, the experimenter pointed to all parts (denominator) one by one. In the continuous version, the experimenter pointed to the colored parts in one single gesture and did the same for all the parts. In the non-gesture version, proportions were just pointed to once. In the transfer task, children have to match a pie chart with the bar representing the same proportion as the pie chart. There were no differences between the three conditions on this task (partial $\eta^2 < .001$), suggesting that children's nonsymbolic skills may require more training sessions to show any improvement. Interestingly, the number of spontaneous continuous gestures children made was related to greater performance in the transfer task, indicating these types of gestures may help overcome children's difficulty with discrete proportions.

DISCUSSION

The goal of this paper was to offer a systematic review of recent intervention and training studies that have used nonsymbolic representations of proportions to promote students' fraction knowledge and relate these results to theories regarding the role of nonsymbolic and symbolic understanding in rational numbers. We identified 19 papers, representing 22 studies, which we grouped in three categories: classroom-based interventions, multiple-session trainings, and single-session training studies.

EDUCATIONAL IMPLICATIONS

Across all categories, the reported results paint an optimistic picture of the malleability of students' fractions skills. All studies that employed numberline tasks were successful in enhancing students' proper fractions skills. In particular, classroom-based interventions suggest that comprehensive interventions are successful in improving fraction skills of fourth-grade students with general math-difficulties or those of fifth- and sixth-grade students who struggle with fractions (Barbieri et al., 2020; Dyson et al., 2020; Fuchs et al., 2016; Jayanthi et al., 2021; Malone et al., 2019). The improvements of these interventions resulted in medium-to-large effect sizes (Cohen's $d > 0.60$) in skills like fraction magnitude understanding and fraction arithmetic when comparing to children's performance in business-as-usual control groups. Notably, there is evidence from one study (Fuchs et al., 2016), that low math achieving students who underwent one of these interventions (*Fraction Face-Off!*) performed equally well in fraction arithmetic problems as typically achieving students after the intervention. Multiple-session training studies showed that trainings that comprised practicing placing nonsymbolic quantities, fractions, or both in a numberline improve students' symbolic fraction knowledge (Braithwaite & Siegler, 2021; Gouet et al., 2020; Kiili et al., 2018; Soni & Okamoto, 2020). On balance, trainings that combined nonsymbolic and symbolic representation of fractions or used only fractions were more successful than those that used exclusively nonsymbolic representations. Finally, single-session training studies (Gunderson et al., 2019; Hamdan & Gunderson, 2017; Sidney et al., 2019; Tian et al., 2021) provide the strongest evidence to advocate for numberline models to represent fractions, as they showed that practicing placing fractions in numberlines for as little as 15 minutes results in greater gains in fraction estimation skills than practicing representing fractions with area models. However, they also underscored the limitations of these models, as one study failed to improve students' estimation skills of improper fractions (Tian et al., 2021). All told, these results advocate the use of numberlines in regular classroom fraction instruction in the late elementary years.

HOW DO TRAINING AND INTERVENTION STUDIES INFORM THE RELATIONSHIP BETWEEN NONSYMBOLIC AND SYMBOLIC PROPORTIONAL ABILITIES?

While basic scientific results are typically translated to the applied setting, applied research can also provide insights into fundamental questions (Rosenberg-Lee, 2018), such as the relationship between nonsymbolic and symbolic proportional ability. Akin to what has been proposed in the whole-number acquisition field (De Smedt & Gilmore, 2011; Mazzocco et al., 2011; Piazza et al., 2010; Rousselle & Noël, 2007), here we identify two proposals for the nature of the relationship between nonsymbolic and symbolic proportional abilities. One proposal posits a core nonsymbolic ability as the foundation for later symbolic skills (the core ability proposal); therefore, difficulties understanding fractions are due to impaired nonsymbolic skills (Lewis et al., 2016; Matthews et al., 2016). The second proposal (the linking proposal) suggests that nonsymbolic skills a generally strong, but need to be linked to symbolic skills; thus, difficulty with fraction learning is not related to impaired nonsymbolic skills but a failure to link nonsymbolic and symbolic proportional skills (Powell, 2018, 2019).

According to the core ability proposal (Lewis et al., 2016; Matthews et al., 2016), the early nonsymbolic proportional ability already present in six-month old infants is the cognitive foundation for our later symbolic fraction skills. Importantly, the precision of individuals' symbolic proportional representations will depend on the precision of nonsymbolic representations, in particular those measured in continuous formats. Supporting evidence is found in recent neuroimaging studies showing that the brain areas responsible for nonsymbolic proportional magnitude processing are also responsible for fraction magnitude processing (for a review, see Rosenberg-Lee, 2021; Wortha et al., 2021). In particular, the intraparietal sulcus, a pivotal region for nonsymbolic and symbolic whole-number magnitude processing (Dehaene et al., 2003; Sokolowski et al., 2017), is active while processing both nonsymbolic and symbolic fraction magnitudes (Mock et al., 2018). A key prediction stemming from this proposal is that improvements in nonsymbolic ability will improve fraction knowledge. To the best of our knowledge, Gouet et al. (2020) is the only training study to date to provide a strong test for this proposal, showing that, indeed, students' symbolic fraction knowledge can be improved by enriching their nonsymbolic proportional skills exclusively. Further evidence is also found in a recent priming study (Szkudlarek & Brannon, 2021): Comparing nonsymbolic proportions before comparing symbolic proportions can improve performance in the symbolic task. Although these studies support the core ability proposal, there is a need for additional studies focused on nonsymbolic only training to show the robustness of these results, ideally in a classroom setting.

An alternative proposal, the linking proposal, suggests that most individuals have strong nonsymbolic proportional skills, particularly for continuous formats. Still, these skills need to be explicitly linked with fraction symbols to improve fraction knowledge (Powell, 2018, 2019). This proposal aims to reconcile the stark contrast between children's nonsymbolic proportional skills and their fraction knowledge; that is, the reasons for children's remarkable continuous nonsymbolic ability but lackluster fraction skills. It also considers the sometimes reported lack of correlation between nonsymbolic and symbolic skills (Matthews & Park, 2021; Rosenberg-Lee et al., 2021). According to this proposal, interventions and training studies will be more successful when they focus on linking the two types of representations, particularly through numberlines. Our review supports this proposal, as the effect sizes for trainings with combined representations were larger than those that only used nonsymbolic ones. The classroom interventions generally involved linking symbolic to nonsymbolic representations (typically numberlines) and were generally successful, further bolstering this claim.

An open question is whether this type of combined training impacts nonsymbolic skills in addition to the symbolic gains of most interest to educators. Interestingly, among the reviewed papers, only one study included an exclusively nonsymbolic comparison task as an outcome measure (Wortha et al., 2020), showing that practicing placing fractions on a numberline improved nonsymbolic skills at the behavioral level but had no effect at the brain level. Assessing the effects of trainings that link symbolic to nonsymbolic on nonsymbolic can helps determine if symbolic gains stem from improving the underlining representations of nonsymbolic proportions, or rather from allowing those existing skills to be activated in the symbolic context.

Another outstanding question regards the contributions of nonsymbolic discrete proportions to fraction knowledge. Although discrete area models may be responsible for some of the misconceptions children have about fractions (Hamdan & Gunderson, 2017), studies show that performance in match-to-sample tasks with discrete proportional stimuli has a stronger relationship to fraction skills than performance with continuous stimuli (Begolli et al., 2020). One possible explanation is that both discrete proportions and symbolic fractions require the same cognitive skills to overcome misconceptions about them. In fact, inhibitory control, the cognitive capacity needed to overcome pre-potent responses (Diamond, 2013), is related to both nonsymbolic (Abreu-Mendoza et al., 2020; Hurst et al., 2022) and symbolic (Avgerinou & Tolmie, 2019; Coulanges et al., 2021; Gomez et al., 2015; Leib et al., 2022) proportional skills. Future studies should examine the effects of trainings and interventions that focus on discrete vs. continuous representations of nonsymbolic proportions on fraction learning and consider the mediating role inhibitory control.

RECOMMENDATIONS FOR FUTURE INTERVENTION AND TRAINING STUDIES

Educational and psychology research can provide complementary information on how best to improve fraction understanding. Yet, each field has its conventions and blind spots. For example, in educational-orientated interventions (classroom-based intervention category), authors rarely reported details about the appearance of numberlines or provided illustrations of the intervention materials. These details are crucial, as experimental psychology research suggests that small features (e.g., hatch marks) can impact numberline effectiveness (Gunderson et al., 2019). On the other handed, training studies which represented paradigms used by experimental psychology lacked detailed descriptions of participant characteristics, information routinely included in educational studies. Relatedly, studies in the single-session training category mostly reported aggregated performance measures rather than detailed performance measures broken out into groups (e.g., Gomez et al., 2015). Future work in this area could benefit from combining quantitative and qualitative measures. An example of mixed-method research on fraction skills can be found in Toledo et al. (this issue), showing the consistencies and discrepancies when comparing experimental tasks vs. written self-reports of strategy used.

One common issue across studies was the various ways in which effect sizes were reported. While some studies reported standard effect size measures (e.g., Cohen's *d*), others reported non-conventional statistics, complicating comparing intervention effectiveness across studies. Here, we suggest that conventional statistics be reported and specifically that the effect size measures for pre- and post-gains should be included, as this metric provides the clearest measure of learning.

Another limitation was the lack of variety in the fraction outcome domains examined. Most studies focused on magnitude processing and procedural skills and used only one measure to assess them. Only one study (Kiili et al., 2018) included a measure of conceptual understanding of fractions, a density task, and notably, gains were weakest in this domain. Thus, the promising results presented here should be tempered with the acknowledgement that these gains may not transfer to other challenging aspects of rational number understanding. Future studies should include measures of conceptual fraction understanding, ideally, with more than one measure for each domain. Finally, when low-achieving students were the focus of the work, studies rarely include a typically achieving group. Including this group affords determining whether children in the intervention groups reached typical performance levels, the ultimate goal of this effort.

CONCLUSIONS

The current study systematically reviewed the findings of intervention and training studies using nonsymbolic representations of proportions, particularly numberlines, to promote students' fraction knowledge. While careful attention must be paid to the specific materials employed and their effectiveness may be limited for more complex aspects of fractions (*e.g.*, improper fractions), results from these papers provide evidence that nonsymbolic representations are a powerful tool for building the foundations of fraction knowledge and may represent an important step in addressing this persistent bottleneck in mathematical knowledge acquisition.

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