CEMeR
Caminhos da Educação Matemática em Revista
2023 • Ano X • v 13 • n. 1 • p. 93-116
ISSN 1983-7399

ARTIGO ORIGINAL/ ORIGINAL ARTICLE

Brazilian math teacher's magnitude representation and strategy use in fraction comparison: a mixed methods study

Representação da magnitude do professor de Matemática brasileiro e uso de estratégia na comparação de frações: um estudo de métodos mistos

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Recebido em 31/03/2022
Aprovado em 31/08/2022

ABSTRACT
Fractions are fundamental in constructing mathematical knowledge, the basis for algebra, and other advanced mathematical content. However, historically, those representations of rational numbers present obstacles for students and teachers. One conceptual base for fractional thinking is magnitude, yet how teachers process fraction magnitudes remains unknown. We investigated Brazilian math educators' knowledge of fraction magnitudes through a convergent parallel mixed-method approach. We collected quantitative data based on teachers' fraction magnitude comparisons, followed by a qualitative task where participants explained their answers on a subset of the comparisons. Participants' magnitude comparison accuracy suggested holistic fraction magnitude processing as rational but not componential distance modulated performance. However, educators' reported strategies revealed many used a flawed Gap strategy, whereby they calculated the difference between the numerator and denominator and selected the fraction with the smallest gap as the larger fraction. As this strategy fails to generalize, teachers' use signals flawed reasoning. Given the relationship between teacher knowledge and student learning, these results have important implications for improving students' rational number outcomes.

Keywords: Fraction Comparison; Mathematics Teacher's Knowledge; Cognitive Psychology.

RESUMO
As frações são fundamentais na construção do conhecimento matemático, base da álgebra e de outros conteúdos matemáticos avançados. No entanto, essas representações de números racionais apresentam obstáculos para alunos e professores. Uma base conceitual para o pensamento fracionário é a magnitude, mas como os professores processam a magnitude de frações é desconhecido. Innvestigamos o conhecimento de educadores matemáticos brasileiros sobre esse assunto, por meio de uma abordagem de métodos mistos. Coletamos dados quantitativos com base nas comparações de magnitude de frações dos professores, seguidas por uma tarefa qualitativa em que eles explicaram suas respostas em uma parte das comparações. A precisão da comparação de magnitude dos participantes sugeriu o processamento de magnitude de fração holística como desempenho racional, mas não modulado por distância componencial. No entanto, as estratégias relatadas pelos educadores revelaram que muitos usaram uma estratégia falha de *Gap*, por meio da qual calcularam a diferença entre o numerador e o denominador e selecionaram a fração com a menor lacuna como a fração maior. Como essa estratégia falha em generalizar, o uso dos professores sinaliza um raciocínio falho. Dada a relação entre o conhecimento do professor e a aprendizagem do aluno, esses resultados têm implicações importantes para melhorar os resultados de números racionais dos alunos.
Palavras-chave: Comparação de Frações; Conhecimento do Professor de Matemática; Psicologia cognitiva.

INTRODUCTION

Historically, fractions have been described as a bottleneck in mathematical concept learning, and their instruction has presented major challenges. This representational format of rational numbers is of fundamental importance in constructing students' mathematical knowledge, as it forms the basis for subsequent, more complex content, and captures an understanding of the relationship between magnitudes, which in turn aids to simplify and solve algebraic equations (BAILEY et al., 2012; BOOTH; NEWTON, 2012; SIEGLER et al., 2012; TORBEYNS et al., 2015).

Previous research has shown that not only do students struggle with this topic, but teachers also encounter obstacles (PINTO, 2011; SERRAZINA; RODRIGUES, 2018; SIEGLER; THOMPSON; SCHNEIDER, 2011; SIEGLER; LORTIER-FORGUES, 2017). In Brazil, teachers' difficulties with fractions have been mostly studied among those teaching at the elementary-school level, and rarely in teachers with mathematics degrees (SANTOS, 2005, CANOVA, 2006; CAMPOS et al., 2006; MAGINA; CAMPOS, 2008; TEIXEIRA, 2008; COSTA, 2011). Therefore, there is a lack of studies on fraction knowledge of teachers with a deeper understanding of mathematics, which could contribute to our understanding of rational number thinking among experts (OBERSTEINER et al., 2013). Moreover, mathematics education research has mostly examined teachers' explicit knowledge about fractions using interviews, leaving out knowledge that teachers themselves do not perceive that they have. In contrast, cognitive psychology examines implicit fraction knowledge using timed computerized tasks, such as fraction comparison and number line tasks. Here, we combined the math education and cognitive psychology perspectives and provided both a qualitative and quantitative examination of math teachers' fraction knowledge. Further, there is considerable evidence that teachers' mathematical knowledge is significantly positively related to student performance gains (HILL; ROWAN; BALL, 2005; DEPAEPE et al, 2015). Therefore, examining teacher's knowledge is important to improve the teaching and learning process of fractions.

Two major factors have been identified in the literature as possible sources of fraction difficulties: 1) the overgeneralization of properties valid for whole numbers but not for fractions (MACK, 1995; FAZIO; SIEGLER, 2011) and 2) the lack of understanding of fraction magnitudes (SIEGLER et al., 2013; SIEGLER; LORTIE-FORGUES, 2015). The overgeneralization has often been referred to as the whole-number bias (NI; ZHOU, 2005) and can be identified in participants' strategies used to compare the magnitude of fractions, such as selecting $\frac{2}{5}$ as greater than $\frac{2}{3}$ because 5 is greater than 3 (MACK, 1995). Beyond comparison tasks, whole-number bias hinders performance on fraction number line tasks and understanding the concept of density for fractions (BRAITHWAITE, 2018; VAN HOOF, 2017). In the context of comparing magnitudes of fractions, this strategy of selecting the larger components could be considered as part of a larger category of mathematically incorrect strategies that only consider the whole-numbers components of fractions that compose fractions instead of considering these numbers as part of a single fraction magnitude. This form of reasoning, called componential thinking, can not only lead to a *larger-component-larger-fraction* strategy but also to a *smaller-component-larger-fraction* strategy. In the current study, we investigate whether math teachers use componential strategies to compare pairs of fractions.

The second factor involved in fraction difficulties is a lack of understanding of fraction magnitudes, which can be observed in children's fraction arithmetic errors. For example, if students understood fraction magnitudes, they would know that $\frac{12}{13} + \frac{7}{8}$ is close to 2. Yet some student select 1 as the answer, suggesting they are applying because of the incorrect procedure $\frac{12+7}{13+8} = \frac{19}{21}$ or even select 19 or 21 reflecting an even more superficial understanding of fraction (VIANNA, 2008). Lack of understanding of fraction magnitude can also be found in fraction comparison tasks. A signature of magnitude understanding are distance effects, that is, the phenomenon where participants are faster and more accurate for pairs of numbers with farer distances (2 vs 9) than nearer ones (8 vs 9) (MOYER; LANDAUER, 1967, NIEDER; DEHAENE, 2009; PINEL et al., 2001).

Evidence for the rational distance effect in fractional comparisons comes from better performance on far problems ($\frac{3}{5}$ $\frac{8}{9}$, $D_{rational} = 0.289$) than on near problems ($\frac{2}{3}$ $\frac{13}{17}$; $D_{rational} = 0.098$). Yet, if participants were focused on componential distance, then they would show the opposite behavioral pattern, that is better performance for far distance between numerators and denominators ($\frac{2}{3}$ $\frac{13}{17}$; $D_{num} = 11$; $D_{den} = 14$) than near distances ($\frac{3}{5}$ $\frac{8}{9}$; $D_{num} = 5$; $D_{den} = 4$). When individuals engage in holistic thinking in fraction comparison tasks, we expect distance effects to be driven by the rational quantities: the greater the distance between the rational magnitude of the fraction, the greater the accuracy and the shorter the reaction time. However, when participants use componential thinking, distance effects could be driven by the numerator or denominator distances, rather than the actual distance (OBERSTEINER et al., 2013; MATTHEWS; LEWIS, 2016; SCHNEIDER; SIEGLER, 2010). While these distance measures are inherently correlated among fractions (ROSENBERG-LEE, 2021), statistical analyses can be employed to disentangle their relative effects. To date, it is unknown whether math teachers are more sensitive to rational distance or componential distance. Thus, a primary aim of this study is to determine if math teachers display componential or holistic thinking in their fraction comparisons.

As examining rational and componential distance effects require statistical analyses of accuracy and reaction time, they do not directly capture other strategies that participants might be employing. Prompting participants to think aloud while solving fraction comparisons is another tool to access strategy use. For example, Smith III (1995) identified strategies that involved using the principle of the numerator and the denominator, transforming a fraction into a decimal number, transforming fractions into closer ones to facilitate further computation, multiplying or dividing the components of fractions to make them near to each other, transforming fractions into equivalent fractions, cross multiplication, and using reference points, like $\frac{1}{2}$ distance and 1 distance. Here, we asked teachers to describe the strategies they used to compare fractions, and then we used content analysis methodology to categorize and interpret them (BARDIN, 2011).

Recent research has also identified other comparison strategies, most notably the Gap strategy. This strategy involves calculating the difference between the numerator and denominator of each fraction (FAZIO et al., 2016; KALRA et al., 2020; GONZÁLEZ-FORTE et al., 2020) and then reasoning that the smaller the difference in the proper fractions, the near they are to the one, the greater the rational magnitude of the number. For example, the Gap distances in the $\frac{3}{4}$ and $\frac{7}{9}$ pair are as follow: gap($\frac{3}{4}$) = 1 and the gap($\frac{7}{9}$) = 2. The Gap strategy would dictate that the fraction $\frac{3}{4}$ would be the largest, as it has the smallest Gap. However, this example also illustrates that the Gap strategy is not always mathematical valid, as in this case, even though the $\frac{3}{4}$ fraction has the smallest gap, it is not the largest fraction, as $\frac{7}{9} > \frac{3}{4}$. To examine whether participants used the Gap strategy, we considered both teachers' explicit strategy use and whether Gap distance effects were evident in participants' performance, indicating implicit use of this strategy.

THE CURRENT STUDY

This study examines the strategies used by Brazilian mathematics teachers when comparing the magnitudes of two fractions. To assess whether teachers are influenced by whole number bias when comparing fractions, participants first completed a timed fraction comparison task (FCT), using the same stimuli employed by Obersteiner et al. (2013)'s study of expert mathematicians. By varying the problem types (i.e. compatible with whole number knowledge or not) we were able to assess whether teachers' performance was driven by a componential or a holistic view of comparing fractions. By varying the distance between stimuli pairs, we assess teachers' implicit knowledge of fraction magnitudes. Combining these manipulations affords examining whether the distance of the components or the actual rational distance drives performance. We complement these assessments of teachers' implicit understanding of fractions magnitudes with an explicit measure of their strategy use. We developed the Justified Fraction Comparison Test (JFCT), where participants explained their reasoning, in writing, for determining which fraction was the largest among a subset of the FCT stimuli. The information obtained from this test allowed us to determine which strategy was the most commonly used by teachers, and how they adapted their strategies depending on the type of fraction pair. Finally, we categorized teachers based on their explicit strategy use and then examined how those groups performed on the FCT task. The overarching goal of the current research is to describe math teachers' implicit and explicit fractional understanding, as well as to identify strategies and reasoning that can inform the development of evidence-based approaches for fraction instruction.

METHODS

Participants

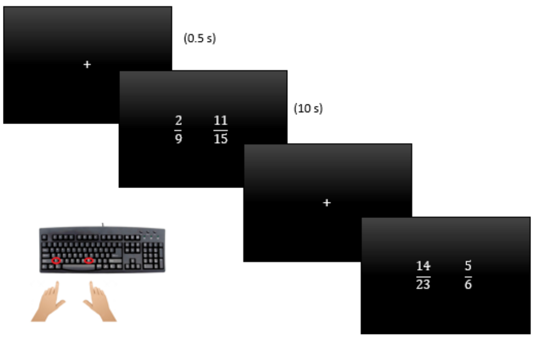
In this study, we had the participation of 49 postgraduate Mathematics teachers with an average age of 33.2 years, of which 31 were female. These teachers work or have worked in the final grades of elementary school or in secondary school, so they do not work directly with introducing fractions in their classes, but instead review the concepts or use it in higher math like algebra. At the time of data collection, all were linked to Postgraduate Programs in the area of Mathematics Education from five different Brazilian higher education institutions. Of the 49 participants, 30 were Master's students, 11 were Ph.D. students and 8 were students enrolled as "special students", that is, not a full-time student of the graduate program but taking some courses (5 Master's special students and 3 Doctoral special students). The tests were administered during 2019.

ASSESSMENTS

Fraction comparison test (FCT)

The test consists of 90 pairs of fractions to be compared. Each pair of fractions was displayed for 10 seconds and replaced with a fixation crosshair for another 0.5 seconds (see Figure 1). Initially, the software used to present the test was PsychoPy [1], with individual data collection, but later Pavlovia[2] was used, which allowed access to the test online, making it possible to collect data from multiple participants simultaneously. The participant's task was to press the key corresponding to the largest of the two fractions presented. They should press the "z" key if the largest fraction was on the left of the screen and they should press the "m" key if the largest fraction was on the right side of the screen. As soon as the participant chose his/her option, by pressing the "z" or "m" key, the screen with a cross would appear to present the next comparison.

Figure 1 – Graphic design of the fraction comparison test.



Source: Elaborated by the authors.

The stimuli of the fraction comparison test were the same as those used by Obersteiner et al. (2013). The test consists of five formats divided into two blocks: one block is composed of fractions pairs with common components (CC), that is, either the same numerators or denominators, and the other block is composed of fractions without common components (WCC). The order of the blocks was counterbalanced across participants [3]. Each format contained 18 comparisons of fractions, totaling 90 comparisons. Within the block, the relationship between the magnitude of the fractions and the magnitude of the components was manipulated relative to whole number bias, such that stimuli could be classified as congruent, incongruent, or neutral).

In comparisons of fractions with CC, those comparisons that have a common denominator are classified as *congruent* (when the numerator of the largest fraction is also the largest numerator). The pairs of fractions that have the common numerator are classified as *incongruent* (that is, when the fraction that has the smallest denominator is the largest fraction).

In comparisons of the block of fractions with different components (WCC) this logic follows. In this block, those comparisons in which the largest fraction has the components (numerator and denominator) greater than those of the smallest fraction are called *congruent*. Comparisons in which the largest fraction has the smallest components are

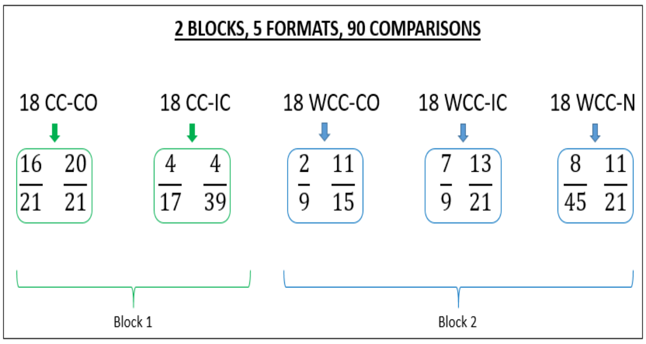
[1] PsychoPy is an open source application that allows you to perform a wide range of experiments in neuroscience, psychology and psychophysics. It is a free alternative written in Python. To prepare this test, PsychoPy version 3.0.3 was used. For access and download: https://www.psychopy.org/. Accessed on 04/10/2019.

[2] Pavlovia is a place for the broad community of behavioral science researchers to perform, share and explore experiments online. Although it was originally designed as a repository and launch pad for PsychoPy experiments, its open architecture makes it possible to support other open source tools, such as jsPsych and lab.js. Information obtained at: https://pavlovia.org/. Accessed on 12/13/2019.

[3] Counterbalance was based on participants' identification number. Participants with odd IDs did block 1 first, followed by block 2. Participants with even IDs did block 2 first.

called *incongruent*. And comparisons in which none of the previous cases occur are called *neutral*, specifically, the largest fraction had the largest numerator and the smallest denominator. Figure 2, provides examples of the problem types.

Figure 2 – Design of the fraction comparison test construction.



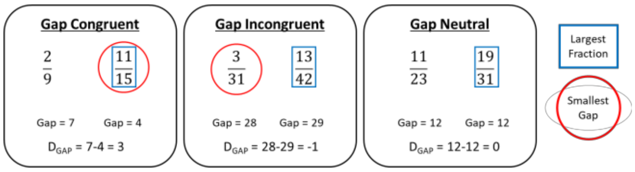
Legend: CC-CO: common components - congruent; CC-IC: common components - incongruent; WCC-CO: without common components - congruent; WCC-IC: without common components - incongruent; WCC-N: without common components - neutral.

Source: Elaborated by the authors.

Within each block, fractions were presented in random order. All comparisons were presented successively in a single session, with a pause between the two blocks. Participants were not instructed to use any specific strategy to solve the comparison problems. The only instruction was to make them as accurate and quick as possible, avoiding mistakes. Initially, four examples of comparisons were presented so that participants could practice the manual response mode. Following the practice trials, the first block was presented.

Motivated by the literature, we also included the strategy Gap as a stimulus category in follow up analyses. This strategy consists of calculating the distance between the components for each fraction (difference between numerator and denominator) and comparing them. The fraction having the smallest distance is then selected as the largest fraction. This strategy follows from the perspective of the parts, as the solver may think that the nearer the numerator is to the denominator the nearer to the whole it is. That is, it is "taking" more parts of the whole. We call the distance between the gaps of each fraction as D_{Gap}.

Figure 3 – Examples of Gap strategies and their classifications. All examples are selected from the WCC-CO stimuli



Source: Elaborated by the authors.

For the classifications of the D_{Gap} variable, the categories of Gap Congruent (D_{Gap}-CO), Gap Incongruent (D_{Gap}-IC), and Gap Neutral (D_{Gap}-N) were used (see Figure 3). In D_{Gap}-CO pairs, the largest fraction had the smallest gap, that is, a positive D_{Gap}; for D_{Gap}-IC pairs the largest fraction had the largest gap and consequently a negative D_{Gap}; and finally, for D_{Gap}-N pairs the two fractions had the same gap and D_{Gap} = 0. Notably, only WCC-CO stimuli can be D_{gap}-IC or D_{gap}-N. WCC-IC and WCC-N stimuli are always D_{Gap}-CO. In the current study, of the 18 WCC-CO pairs, 12 are classified as D_{Gap}-CO, 4 as D_{Gap}-IC, and 2 as D_{Gap}-N. Therefore, the Gap strategy analyses will be restricted to the WCC-CO format.

Justified fraction comparison test (JFCT)

This test was designed to assess participants' explicit strategy use when solving fraction comparisons. Seven pairs of fractions were selected for this task in which participant compare and choose which was the largest fraction among the two, and then explain how they arrived at that conclusion.

The choice of the largest fraction and the explanation for such decision were tasks presented in different screens of the questionnaire implement in Google Forms. Thus, the participant no longer saw the fraction pair when they had to explain why they made their choice. A practical example was presented before starting the comparisons and the advice given was not to use paper, pencil, calculator, or any other materials to select which option they considered as the largest number.

One pair with CC and two pairs from each of the other categories formed with different components (WCC-CO, WCC-IC, WCC-N) were selected for the stimuli. To select the two stimuli of each category, a pair that had a greater absolute distance between the fractions and a pair that had this smallest distance was chosen. The pairs of fractions, their characteristics, and presentation order of showing in Table 1.

Table 1 – Fraction pairs of the Justified Fraction Comparison Test and its characteristics

Pair of fractions to be compared		Absolute distance	Component category
$\frac{6}{13}$	$\frac{6}{47}$	0.33	CC - IC
$\frac{13}{42}$	$\frac{3}{31}$	0.21	WCC-CO
$\frac{6}{17}$	$\frac{5}{8}$	0.27	WCC-IC
$\frac{8}{11}$	$\frac{12}{23}$	0.21	WCC-IC
$\frac{11}{23}$	$\frac{19}{31}$	0.13	WCC-CO
$\frac{9}{28}$	$\frac{11}{27}$	0.09	WCC-N
$\frac{11}{18}$	$\frac{8}{29}$	0.34	WCC-N

Legend: CC-IC: common components - incongruent; WCC-CO: without common components - congruent; WCC-IC: without common components - incongruent; WCC-N: without common components - neutral.

Source: Elaborated by the authors.

DATA ANALYSIS PROCESS

This work is classified as a convergent parallel mixed method design (CRESWELL; CRESWELL, 2018) as it is composed of data collection instruments that require quantitative and qualitative analyses. Ideally, qualitative analyzes and quantitative analyses should be in dialogue and provide complementary information.

For that quantitative data analysis, we first used analyses of variance (ANOVA) to examine the accuracy and reaction time differences across the blocks and congruency conditions in the Fraction Comparison Test (FCT). Then, we used logistic and linear mixed models to quantify how the different types of distances (rational, numerator, denominator, and gap distances) modulated participants' accuracy and reaction times, respectively. For visualization purposes, we fitted simple linear regression models over the averaged accuracy and reaction times to illustrate the relation between performance and the different types of distances.

For the qualitative data analysis, we applied content analysis (BARDIN, 2011) to uncover and interpret the information obtained in the Justified Fraction Comparison Test (JFCT), by analyzing the justifications given by the participants regarding their choices. For the content analysis, we followed the steps proposed by Bardin (2011): pre-analysis (organization of the material), exploration of the material (selecting categories and counting items in each category), and treatment of results (inference and interpretation).

Finally, to examine the relationship between performance on the quantitative FCT and the type of strategies participants reported in the qualitative JFCT, we classified participants into three groups based on the strategy most used by them. Then, we descriptively compared the performance across the different blocks and congruency conditions of the fraction comparison test.

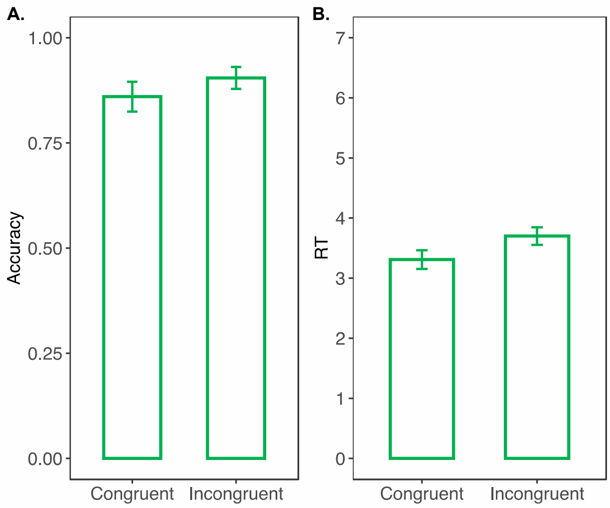
RESULTS

Overall performance across the different fraction types

Common components

To investigate whether participants' accuracy and reaction times differed across the congruent and incongruent trials, we performed one-way ANOVAs with Congruency (Congruent and Incongruent) as the within-subject factor, for accuracy and reaction time. For accuracy, there was no main effect of Congruency ($F(1,47) = 2.31$, $p = .135$, eta = .011), as performance was high in both conditions (CC-CO = 86% vs. CC-IC = 90%, Figure 4A). By contrast, for reaction times, there was a main effect of Congruency ($F(1,45) = 14.66$, $p < .001$, eta = .036), indicating that participants were faster on congruent trials (mean = 3.31, SD = 1.06) than incongruent ones (mean = 3.70, *SD* = 0.99, Figure 4B).

Figure 4 – Effect of Congruency in accuracy and reaction time in CC fractions

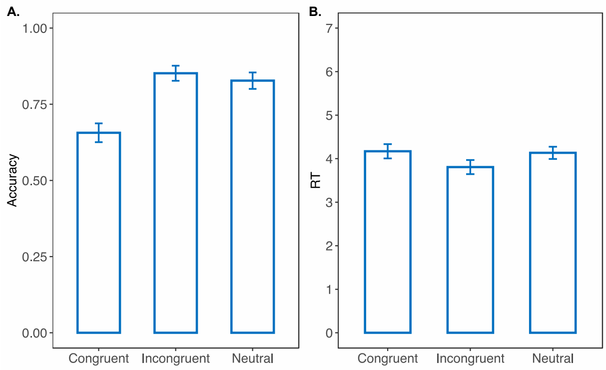


Source: Research data.

Without common components

For the WCC problems, we found a significant main effect of Congruency, between the three conditions, Congruent, Incongruent and Neutral ($F(2,94) = 43.36$, $p < .001$, eta = .173). Follow-up paired *t*-tests indicated that participants had lower accuracy in the congruent (65%) trials relative to incongruent (85%, $t(47) = 6.99$, $p < .001$) and neutral (83%, $t(47) = 7.89$, $p < .001$) trials (Figure 5A), which did not differ from each other ($t(47) = 1.35$, $p = .182$). For reaction times there, there was also a main effect of Congruency ($F(2,90) = 7.218$, $p = .001$, eta = .024, Figure 5B). Follow-up paired *t*-tests indicated that participants were faster in the incongruent condition than the congruent ($t(45) = 4.09$, $p < .001$) and neutral trials ($t(45) = 2.79$, $p = .007$), which did not differ from each other ($t(45) = 0.34$, $p = .736$).

Figure 5 – Effect of Congruency in accuracy and reaction time in WCC fractions



Source: Research data.

Distance effects

Following Obersteiner et al. (2013), Table 2 and 3 present a summary of the results from the logistic and linear mixed models for the three distance effects reported there (numerator, denominator, and rational distance) and for each of the five different types of fractions with accuracy and reaction times as the dependent variables, respectively. Figure 6 and 7 illustrates the simple linear regression models over the averaged accuracy and reaction times, respectively, for the effect of the different types of distances on performance.

Accuracy

To examine how each of the different distances (*i.e.*, rational, numerator, and denominator) modulated participants' accuracy, we performed generalized logistic mixed-effects models with binomial distributions, for each of the different distance metrics as fixed effects and participant as a random effect.

CC Block. For this block, we only examined the distance effects of the component that varied: for the congruent trials, that was numerator distance, and for the incongruent trials, the denominator distance. Notably, none of the distance metrics modulated participants' accuracy on the congruent or the incongruent trials.

WCC Block. For the *congruent* trials of this block, all three distances modulated teachers' accuracy. As expected, the effects of the numerator and rational distances were positive. Participants were more accurate in large-distance trials than small-distance ones). The effect of the denominator distance was negative, indicating that participants had lower accuracy for larger denominator distances, as previously reported (OBERSTEINER et al., 2013). For the *incongruent* trials, only rational distance positively modulated participants' performance. Finally, for the *neutral* trials, all distances positively modulated participants' accuracy.

Table 2 – Logistic mixed models for the distance effects and each type of fractions with accuracy as the dependent variable

Source: Research data.

Block	Condition	Predictor	R2	B	SE	z	Sig.
Common Components	Congruent	Numerator Distance	0.00	0.00	0.05	-0.01	0.996
		Denominator Distance	-	-	-	-	-
		Rational Distance	0.00	-0.68	1.61	-0.43	0.67
	Incongruent	Numerator Distance	-	-	-	-	-
		Denominator Distance	0.01	0.03	0.02	1.26	0.208
		Rational Distance	0.00	-0.23	1.56	-0.15	0.881
Without Common Componets	Congruent	Numerator Distance	0.01	0.15	0.06	2.58	.010*
		Denominator Distance	0.02	-0.92	0.03	-3.45	.001**
		Rational Distance	0.03	4.14	0.99	4.19	< .001***
	Incongruent	Numerator Distance	0.00	0.01	0.05	0.20	0.841
		Denominator Distance	0.00	-0.05	0.04	-1.20	0.229
		Rational Distance	0.02	4.88	1.80	2.71	.007**
	Neutral	Numerator Distance	0.01	0.08	0.04	1.97	.049*
		Denominator Distance	0.03	0.06	0.02	3.85	< .001***
		Rational Distance	0.04	4.23	1.07	3.96	< .001***

To determine whether componential distance processing explained performance over and above rational distance, we compared models that included rational distance and any significant componential distance (comparison model) against a model that only included rational distance (base model) using likelihood ratio tests. For congruent trials, the comparison model included the rational distance and the two componential distances. The comparison model only marginally improved the AIC fit index (chi-squared (2) = 4.84, *p* = .089, Table 3). For neutral trials, the comparison model also included the three types of distances. In this case, the comparison model did not improve the AIC fit index (chi-squared (2) = 1.76, *p* = .414, Table 4). We did not model incongruent trials as only rational distance predicted their performance.

Table 3 – Predicting accuracy with logistic mixed models for WCC-CO trials

Predictor	Base Model		Comparison Model	
	z	*p*	*z*	*p*
Intercept	-0.159	0.874	-0.287	0.774
Rational Distance	4.186	<.001	2.749	0.006
Numerator distance			1.396	0.163
Denominator distance			-1.685	0.092

Source: Research data

Table 4 – Predicting accuracy with logistic mixed models for WCC-N trials

Predictor	Base Model		Comparison Model	
	z	*P*	*z*	*p*
Neutral				
Intercept	4.015	<.001	4.173	<.001
Rational Distance	3.955	<.001	0.879	0.379
Numerator Distance			0.372	0.710
Denominator Distance			1.330	0.184

Source: Research data.

Reaction times

To examine the effects of each of the different distances (*i.e.*, rational, numerator, and denominator) on participants' reaction time, we performed generalized linear mixed-effects models for each of the different distance metrics as the fixed effect and participant and item as random effects. We were able to include the item as a random effect in this analysis the number of data points per item varied by item by item difficulty.

CC Block. As with accuracy, we did not examine the effects of the denominator and numerator distances for the *congruent* and *incongruent* trials, respectively. For the *congruent* trials, only rational distance negatively modulated participants' reaction times. By contrast, the incongruent trials' reaction times were both negatively modulated by the rational and numerator distances, indicating that reaction times of trials with larger distances were faster than those of smaller distances.

WCC Block. For the *congruent* trials of this block, the rational distance modulated teachers' reaction time negatively, indicating that participants had shorter reaction times for larger distances. By contrast, the denominator distance modulated positively. For the *incongruent* trials, only rational and numerator distances negatively modulated participants' reaction times. Finally, rational and denominator distances negatively modulated participants' reaction time for the *neutral* trials.

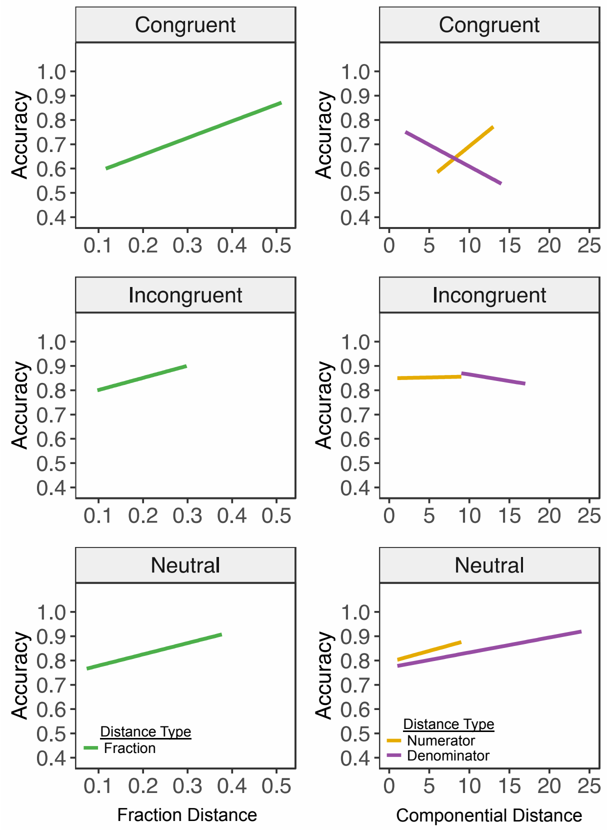
Table 5 – Linear mixed models for the distance effects and each type of fractions with reaction times as the dependent variable

Block	Condition	Predictor	R2	B	SE	z	Sig.
Common Components	Congruent	Numerator Distance	0.00	-0.02	0.02	-0.81	.433
		Denominator Distance	-	-	-	-	-
		Rational Distance	0.01	-1.80	0.61	-2.96	.003**
	Incongruent	Numerator Distance	-	-	-	-	-
		Denominator Distance	0.01	-0.02	0.01	-2.27	.038*
		Rational Distance	0.01	-1.44	0.67	-2.16	.047*
Without Common Components	Congruent	Numerator Distance	0.00	-0.06	0.08	-0.82	.424
		Denominator Distance	0.01	0.08	0.03	2.24	.040*
		Rational Distance	0.02	-2.71	0.97	-2.78	.014*
	Incongruent	Numerator Distance	0.02	-0.11	0.04	-2.70	.016*
		Denominator Distance	0.01	-0.09	0.05	-1.90	.076
		Rational Distance	0.03	-5.12	1.51	-3.38	.004**
	Neutral	Numerator Distance	0.02	-0.10	0.05	-2.07	.056
		Denominator Distance	0.04	-0.05	0.01	-3.82	.001**
		Rational Distance	0.05	-4.28	0.77	-5.61	< .001***

Source: Research data.

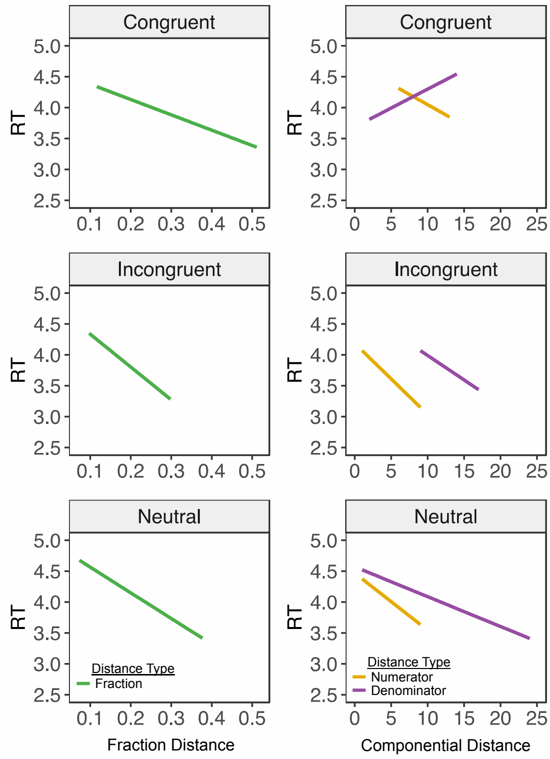
Finally, we examined whether componential distances explained variations in reaction times above and beyond rational distance by contrasting a base model with only the rational distance and a comparison model, which also included the significant componential distances. We performed these comparisons for the following trials: CC-IC and the three types of trials of the WCC block. For the WCC-IC trials, reaction times were modulated by both the rational and numerator distances (chi-squared (2) = 11.96, p = .002, Table 6). For all the other types of trials, componential distances did not explain any variation beyond the one already explained by rational distance (CC-IC, chi-squared (1) = 1.43, p = .231; WCC-CO, chi-squared (1) = 2.00, p = .157; WCC-N, chi-squared (2) = 0.65, p = .720).

Figure 6 – Accuracy for without common component (WCC) trials showing distance effects (fraction, numerator, denominator) for each trial type



Source: Research data.

Figure 7 – Reaction times (RT) for without common component (WCC) trials showing distance effects (fraction, numerator, denominator) for each trial type.



Source: Research data.

Table 6 – Predicting reaction time with for linear mixed models for performance in the WCC-IC trials

Predictor	Base Model t	Base Model p	Comparison Model t	Comparison Model p
Intercept	13.720	<.001	11.456	<.001
Rational Distance	-3.384	0.004	-4.286	<.001
Numerator Distance			-2.986	0.010
Denominator Distance			0.028	0.978

Source: Research data.

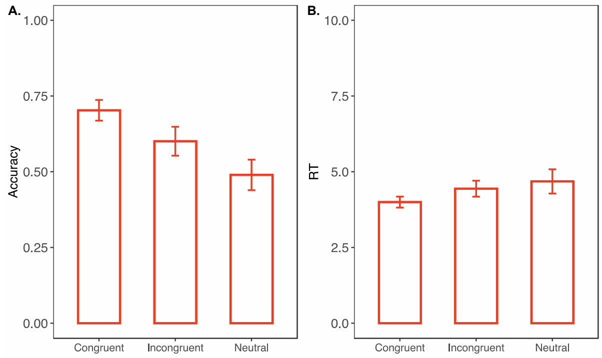
GAP EFFECTS

To examine the effects of Gap congruency on participants' responses and reaction times in WCC-CO trials, we conducted one-way

ANOVAs with Gap congruency (congruent, incongruent, neutral) as within-subject factor, separately for accuracy and reaction times. We only considered these WCC-CO trials, as Gap distance can be congruent, incongruent, and neutral for them, but it is always congruent or neutral for WCC-IC and WCC-N trials. The ANOVA with accuracy as the dependent variable indicated a main effect of Gap classification ($F(2,94) = 7.68$, $p < .001$, $eta = .074$). Post-hoc *t*-tests indicated that teachers were more accurate in Gap-CO trials than Gap-IC trials (stats), and in turn, (marginally) more accurate in Gap-IC than Gap-N trials (Figure 8A). The ANOVA with reaction times as the dependent variable showed no main effect of Gap congruency ($F(2,64) = 2.32$, $p = .106$, $eta = .028$, Figure 8B).

Next, we examined the role of Gap distance as a continuous factor that could range from positive to negative (in the case of Gap-IC stimuli). We performed logistic and linear mixed models with Gap distance as the fixed factor and accuracy and reaction times as the dependent variable, respectively. Participants were more accurate ($B = 0.094$, $SE = 0.024$, $p < .001$) and faster ($B = -0.063$, $SE = 0.028$, $p = .041$) the larger the Gap distance between the two pairs of fractions. For accuracy, both rational and Gap distance independently modulated participants' correct responses (Table 7 top part). However, when we introduced the rational distance to the reaction time model, rational distance had a marginal effect on reaction time, and there was no main effect of Gap distance (Table 7 bottom part).

Figure 8 – Accuracy on WCC-CO trial by Gap classification



Source: Research data.

Table 7 – Predicting accuracy with logistic mixed effect models (top) and reaction time with linear mixed models (bottom) WCC-CO trials.

	Base Model		Comparison Model	
Predictor	*z*	*p*	*z*	*p*
Accuracy				
Intercept	3.94	< .001	0.45	.654
Gap Distance	3.94	< .001	2.13	.030*
Rational Distance	-	-	2.78	.006**
	t	*p*	*t*	*p*
Reaction times				
Intercept	24.28	< .001	17.79	< .001
Gap Distance	-2.23	.041*	-1.25	.232
Rational Distance	-	-	-1.95	.071

Source: Research data.

JUSTIFICATION FRACTION COMPARISON TEST - QUALITATIVE DATA

Results of the JFCT revealed generally strong performance (85.38%), with half of the participants answering all questions correctly. Table 8 lists the performance on each comparison and Figure 9 provides an illustration on how the different categories were distributed in each comparison.

Table 8 – TCFJ fraction pairs and strategy characteristics

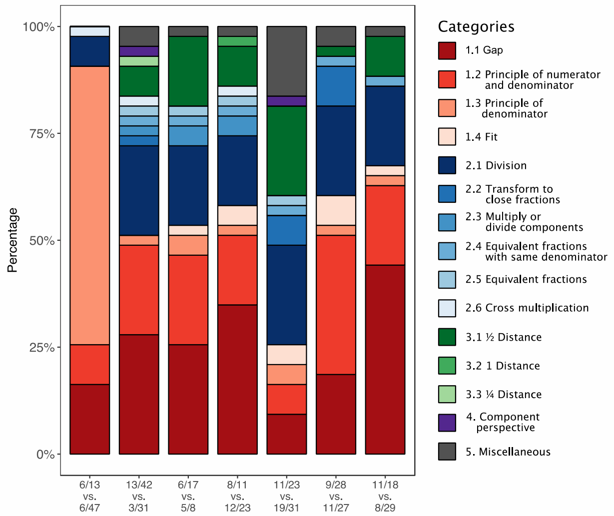
Pair of fractions compared	Average accuracy	Component category	D_{Gap} value	Gap category	Most common strategies
(a) $\frac{6}{13}$ $\frac{6}{47}$	95.35%	CC - IC	$D_{Gap}(\frac{6}{13}\ \frac{6}{47}) = 34$	D_{Gap}-CO	**PP: denominator principle** (65.9%)
(b) $\frac{13}{42}$ $\frac{3}{31}$	83.72%	WCC-CO	$D_{Gap}(\frac{13}{47}\ \frac{3}{31}) =- 1$	D_{Gap}-IC	**TP: Division** (26.5%) PP: Gap (20.6%) PP: num. & den. principle (20.6%)
(c) $\frac{6}{17}$ $\frac{5}{8}$	90.70%	WCC-IC	$D_{Gap}(\frac{6}{17}\ \frac{5}{8}) = 8$	D_{Gap}-CO	**PP: Gap** (30.8%) PP: num. & den. principle (17.9%)
(d) $\frac{8}{11}$ $\frac{12}{23}$	90.70%	WCC-IC	$D_{Gap}(\frac{8}{11}\ \frac{12}{23}) = 8$	D_{Gap}-CO	**PP: Gap** (40.0%) TP: division (17.5%)
(e) $\frac{11}{23}$ $\frac{19}{31}$	69.77%	WCC-CO	$D_{Gap}(\frac{11}{23}\ \frac{19}{31}) = 0$	D_{Gap}-N	**RPP: ½ distance** (37.9%) TP: division (31.0%)
(f) $\frac{9}{28}$ $\frac{11}{27}$	79.07%	WCC-N	$D_{Gap}(\frac{9}{28}\ \frac{11}{27}) = 3$	D_{Gap}-CO	**PP: num. and den. principle** (34.4%) PP: Gap (21.9%)
(g) $\frac{11}{18}$ $\frac{8}{29}$	88.37%	WCC-N	$D_{Gap}(\frac{11}{18}\ \frac{8}{29}) = 14$	D_{Gap}-CO	**PP: Gap** (48.6%) TP: division (18.9%)

Legend: CC-IC: common components - incongruent; WCC-CO: without common components - congruent; WCC-IC: without common components - incongruent; WCC-N: without common components - neutral. PP: Part perspective, TP: Transformation perspective, RPP: Reference point perspective.

Source: Elaborated by the author

The justifications provided by the participants were analyzed according to the content analysis methodology (BARDIN, 2011): pre-analysis (organization of the material), exploration of the material (selecting categories and counting items in each category), and treatment of results (inference and interpretation). Initially, we applied the categories developed by Smith III (1995), which consists of the (1) Parts, (2) Transformation, (3) Reference Point, and (4) Component perspectives. From these categories, we created further subcategories. In the Part Perspective (PP) category, we identified subcategories: (1.1) Gap, (1.2) Principle of the numerator and the denominator, (1.3), Principle of the denominator, and (1.4) Fit. In the Transformation Perspective (TP) category we have (2.1) Division, (2.2) Transform to close fractions, (2.3) Multiply or divide the components of a fraction to get closer to the other, (2.4) Equivalent fractions with the same denominator, (2.5) Equivalent fractions, and (2.6) Cross multiplication. For the Reference point perspective (RPP), we developed three subcategories: (3.1) ½ distance, (3.2) 1 distance, and (3.3) ¼ distance. Category 4, the Component Perspective (CP), did not have any subcategories. Category 5, Miscellaneous, comprised responses classified as (5.1) No strategy provided, (5.2) Blank responses, and (5.3) Unable to classify. Table 9 explains each category and subcategory, and it provides an example for each, in addition to the percentage of responses employing these categories across all trials.

Figure 9 – Percentage of teachers who used each of the different strategy categories for each comparison of the JFCT.



Source: Research data.

Table 9 – Categorization of the comparison strategies used by the research participants.

Subcategories	Examples	Percentage
(1) Perspective of the parts (PP): the participant uses the consequences of understanding the fraction as part of a whole to compare the fractions.		**57.7%**
(1.1) Gap: the difference or distance between the numerator and the denominator of the fraction. The subject selects that the smaller distance between the components as the larger the fraction.	"As a strategy, I looked at the difference between the numerator and denominator. I opted for the smallest difference between them to choose the largest fraction. For example, 6/7 and 6/15, on the 1st the	25.6%

	difference is one and on the second the difference is 9, so I chose the 1st as a larger fraction." (P07, [6/13 6/47])	
(1.2) Principle of the numerator and the denominator: the higher the numerator the greater the fraction, as the numerator is understood as the number of parts it is considering of the whole that was divided, and the greater the denominator, the smaller the fraction, as the denominator is understood as the number of parts that the whole was divided.	"In this case, it can be seen that the numerator of the second is greater than that of the first and its denominator is lower than that of the first. Thus, its value is greater than that of the first." (P22, [9/28 11/27])	17.0%
(1.3) Principle of the denominator: the larger the denominator the smaller the fraction, that is, the more parts I divide the whole, the smaller they are.	"As the fractions have the same numerator, the largest has the lowest denominator. So I have bigger parts of a whole." (P01, [6/13 6/47])	12.1%
(1.4) Fit: is the number of times the numerator "fits" within the denominator. The subject selects the lowest FIT number as the largest fraction.	"I thought that in the first division we have 8 divided by 11, a value (11) much less than double 8, and in the other fraction 12 divided by 23, almost double (12), so I think the biggest it	3.0%

	would be 8\11." (P35, [8/11 12/23])	
(2) Transformation perspective (operations, manipulations) (TP): the participant modifies one or both fractions to transform it(s) into something more familiar.		**26.3%**
(2.1) Division: the transformation of fractions into decimal numbers by dividing the numerator by the denominator using the division algorithm, in Brazil the key algorithm is used to position them on the numerical line.	"Looking at the numerators and denominators, imagining the division between the two and transforming them into decimals." (P16, [13/42 3/31])	17.7%
(2.2) Transform to close fractions: find close fractions with terms close to one of the fractions.	"If we had 9/27 and 11/27, it is easy to see that 11/27 is greater. As the fractions are 9/28 and 11/27 and 9/28 is less than 9/27, I realized that 11/27 and greater." (P01, [9/28 11/27])	2.6%
(2.3) Multiply or divide the components of a fraction to get closer to the other.	"I divided the first fraction by 2, having an idea that it would be 6.5/21. Imagining that the denominators are "close", I came to the conclusion that the first is bigger." (P47, [13/42 3/31])	2.0%

(2.4) Equivalent fractions with the same denominator: obtain fractions equivalent to the two fractions given with common denominators. For this, the technique of the least common multiple is used.	"I reduced it to the same denominator and compared the values." (P29, [6/17 5/8])	2.0%
(2.5) Equivalent fractions: obtain a fraction equivalent to one of the fractions that have the terms closest to the other, to obtain fractions with the closest terms.	"Using equivalent fractions (approximation attempt)." (P20, [6/17 5/8])	1.3%
(2.6) Cross multiplication: multiply the top of the left side with the bottom of the right side and equal it with the bottom of the left side multiplied with the top of the right side.	"As the two fractions have different numerators and denominators, I multiplied the means and the extremes. If the product of the media is greater, the first fraction will be greater, if the product of the extremes is greater, the second fraction will be the largest." (P1, [13/42 3/31])	0.7%
(3) Reference point perspective (RPP): the subject compares the magnitude of the fraction to familiar points on the numerical line.		**10.7%**
(3.1) ½ distance	"Again, there was a comparison with the fraction 1/2. The second fraction is greater than 1/2 and the first is less than 1/2. Thus, the second fraction is greater than the first." (P22, [6/17 5/8])	10.1%
(3.2) 1 distance	"Near to the whole." (P40, [8/11 12/23])	0.3%
(3.3) ¼ distance	"The first fraction is greater than 1/4 while the second is less." (P15, [13/42 3/31])	0.3%
(4) Component perspective (CP): the subject is concerned with the magnitude of each component of the fraction separately. Without thinking about the meaning of the fraction, the subject perceives the fraction as a number on top of a bar and another number below. Look only at the whole numbers that compose it and evaluate which one has the largest components. (Can consider only the numerators or only the denominators of the fractions).		**0.7%**
	"The numerator and denominator are smaller in the first fraction. That is why it is the first." (P28, [11/23 19/31])	
(5) Miscellaneous		**4.6%**
(5.1) No strategy was provided	"I noticed that the difference between the numerator and the denominator are the same. (I think my strategy is not convenient)." (P04, [11/23 19/31]) "They are equivalent, by	2.0%

	the chosen strategy. In other words, the division should be the same, but I don't know if this is true, I had doubts." (P07, [11/23 19/31]) "As the reasoning of the previous questions does not apply I "kicked" the answer." (P46, [11/23 19/31])	
(5.2) Blank responses		2.0%
(5.3) Unable to classify	"For the first is proportionally greater." (P21, [13/42 3/31])	0.7%

Source: Elaborated by the author.

The three most used perspectives (PP: Gap, TP: division, PP: numerator and denominator principle) appeared in more than 60% of the participants' answers. The Component Perspective (CP) was used in only 0.66% of the responses, confirming the results obtained in the fraction comparison task, namely, that fraction components alone (i.e. numerators and denominators) did not influence participants' decisions. Table 7 lists the accuracy for each problem separately, their properties of the as well as the most used perspectives.

In the first pair of fractions presented (a) $\frac{6}{13}$ $\frac{6}{47}$, where accuracy was highest, both fractions had the same numerator. Consequently, the strategy most used by the participants was the PP: Denominator principle. Among the remaining WCC stimuli, we expected the highest performance on WCC-CO stimuli, but as in the FCT, these were among the lowest performing condition (69.77% and 83.72%). This pattern of results provides further evidence that participants were not basing their selections on the larger components.

Note that among the WCC pairs of fractions with the highest success rate (pairs (c), (d), and (g)), the most used strategy was PP: Gap. In the other three pairs, the comparison strategies varied. In pair (b) $\frac{13}{42}$ $\frac{3}{31}$ the strategy they used most was TP: Division (26.5%). However, PP: Gap was still widely used (20.6%). For the pair (e) $\frac{11}{23}$ $\frac{19}{31}$ the strategy was guided by the RPP: ½ distance (37.9%). Finally, for the pair (f) $\frac{9}{28}$ $\frac{11}{27}$ the most used was the PP: Numerator and denominator principle (34.4%), but PP: Gap was used 21.9% of the time.

Overall, the qualitative data show little use of component-based strategies, some use of magnitude (reference point) and procedure (transformation perspective), but mostly Gap strategy. To better understand the use of this strategy we categorized the pairs of fractions considering the distance of the Gaps and performed new analyses.

ANALYZING THE USE OF THE GAP STRATEGY

The PP: Gap strategy is the only strategy used by the participants that is not mathematically valid for all fractions pairs. The smallest difference between the components will not always represent the largest fraction. Take the $\frac{3}{4}$ $\frac{7}{9}$ pair as an example. The different between the numerator and denominator for $\frac{3}{4}$ is 1 and the for $\frac{7}{9}$ it is 2. Thus, $\frac{3}{4}$ has the smallest Gap but is not the largest fraction, as $\frac{7}{9} > \frac{3}{4}$.

As illustrated in Table 8, except the pair with CC, in the three problems with the highest accuracy ((c), (d), and (g)), the most used strategy (Gap) worked very well, since in the three cases the D_{Gap} is congruent. Notably, for these same pairs, the Gap distances are large (D_{Gap} ($\frac{6}{17}$ $\frac{5}{8}$) = 8, D_{Gap} ($\frac{8}{11}$ $\frac{12}{23}$) = 8, and the D_{Gap} ($\frac{11}{18}$ $\frac{8}{29}$) = 14), making it easier to check the smallest difference between the numerator and the denominator. This was reasoning is evident in participant P04's justification: *"The largest fraction is the one with the smallest difference between the numerator and denominator (where the numerator is smaller than the denominator)."* (P04,[$\frac{8}{11}$ $\frac{12}{23}$], PP: Gap).

Given that PP: Gap strategy worked very well for the three pairs presented above, what made this strategy not also the most used in other pairs? When the participants who used PP: Gap were faced with the pairs presented in items (b) $\frac{13}{42}$ $\frac{3}{31}$ and (e) $\frac{11}{23}$ $\frac{19}{31}$, some participants realized that their strategy was not valid, because in these questions the fraction with the smallest Gap was not the greater fraction and the Gap of the two fractions were equal, respectively.

On (b) $\frac{13}{42}$ $\frac{3}{31}$, the fact that PP: Gap was not the most used justification is likely because it does not apply to this pair. In this case D_{Gap} ($\frac{\mathbf{13}}{\mathbf{42}}$ $\frac{3}{31}$) $= -1$, that is, the smallest Gap does not represent the largest fraction. For some participants, applying the Gap strategy, led them to error.

> I compared the numerator and denominator of each fraction, and I chose the one that has the smallest difference between numerator and denominator. As in the first fraction, the difference is 29 and in the second the difference is 28, so the second fraction is greater than the first. (P48) (PP: Gap)

Interestingly, many participants report using PP: Gap and still got the question right (20.59%). This can be explained by the fact that the Gaps are very large numbers (29 and 28) and close (the distance between the Gaps is -1), which may have confused them.

Analyzing the comparison (e) $\frac{11}{23}$ $\frac{19}{31}$, which had the lowest accuracy rate, it can be seen that the $D_{Gap} = 0$, that is, the two fractions have the same Gap = 12. Some participants who had consistently used PP: Gap to this point realized the issue with their strategy and noted its inadequacy, but did not look for another strategy: *"I realized that the difference between the numerator and the denominator are the same (I think my strategy is not appropriate)."* (P04); *"They are equivalent, according to the chosen strategy. In other words, the division should be the same, but I don't know if this is true, I had doubts."* (P07); *"Since the reasoning of the previous questions doesn't apply, I guessed at the answer."* (P046).

Others realized that the PP: Gap did not help them and then changed their strategy, most commonly to RPP: $\frac{1}{2}$ distance and the TP: Division:

> As the differences between the numerator and the denominator of each fraction are equal, I did the calculation a mental division to the first decimal place, until I found that the first one resulted in 0.4... and the second in 0.6... So the second fraction is bigger. (P048) (TP: division).

While some participants realized the issue with PP: Gap, others maintained its utility for teaching in teaching fractions: *"I think it is a solution to explain to a child that the near the numbers, the greater the division."* (P23) (PP: Gap).

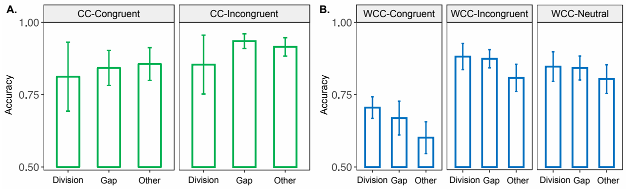
Finally, in item (f) $\frac{9}{28}$ $\frac{11}{27}$ the most used was PP: Numerator and denominator principles. This pair was presented after the pair whose D_{Gap} was neutral, that is, some participants had just realized the non-validity of PP: Gap for all comparisons. Using another strategy, the most convenient for the participants was PP: principle of numerator and denominator, which in this case was very useful, as these are fractions of the WCC-N case, that is, *"Among the fractions, the chosen one has the lowest denominator and the highest numerator."* (P06) (PP: principle of numerator and denominator).

In general, it is observed that the most used strategies were influenced by the meaning of the fraction as part-whole, being the perspectives PP: Gap and PP: Numerator and denominator principles. Besides these, the fraction transformation to the decimal representation was also significantly chosen, indicating the use of the division algorithm, better known in Brazil as the key algorithm, and the reference point perspective, more significantly the RPP: ½ distance. Of these four perspectives, there is only one that will not always provide the participant with a correct answer, is PP: Gap. However, the very exercise of comparing fractions made some participants realize the lack of validity of this perspective for all cases.

PERFORMANCE BY GROUPS OF STRATEGISTS

To analyze performance on the fraction comparison task depending on the type of strategies participants used, we classified participants into three groups based on the strategy most used by them. Specifically, we looked for participants using the same strategy more than 3 times. From this approach, we identified 3 groups: the Gap strategist group (12 participants), the Division strategist group (8 participants), and the Other group, which comprised participants that did not fall into the other groups.

Figure 10 – Performance by groups of strategists



Source: Research data.

In pairs with CC-CO (Figure 10A), the best performance was among those who used other strategies (85.6%), followed by the Gap strategists (84.3%) and by Division strategists (81.3%). There was a small change for CC-IC pairs, in which Gap strategists were better (93.5%), followed by those who used other strategies (91.5%) and by Division strategists (85.4%). For all WCC cases, Division strategists outperformed the two groups (Figure 10B). In particular, when averaging all cases, the Division strategists (81.1%) outperformed the Gap strategists (79.5%), which in turn outperformed Other strategists (73.8%).

DISCUSSION

In this study, we examined how postgraduate mathematics teachers process fraction magnitude, combining the traditions of cognitive psychology and math education. Specifically, we collected quantitative data from a fraction comparison task assessing the implicit knowledge of teachers. We complemented this quantitative data with qualitative content analyses of participants' explanations of their answers on a subset of the fraction comparisons. Results from the fraction comparison task indicated that teachers' performance was modulated by the rational distance between the pairs of fractions, instead of the distance between their components, suggesting that teachers engaged in holistic thinking of fractions instead of a componential thinking. We also found evidence that Gap distance was driving performance on more difficult WCC problems. Results from the qualitative analyses add nuance to the quantitative analyses. We found that participants used a variety of strategies to compare fractions, which range from reasoning about fractions using a part-whole perspective to manipulating fraction components. Notably, one strategy that stood out from the rest was the Gap strategy, which was used often, despite its lack of applicability for all problems. In summary, the combination of qualitative and quantitative tests showed that mathematics teachers do not think about fractions componentially; however, they use different strategies, not necessarily consistent with a purely holistic view of fractions.

Whole-number bias or componential thinking

Past studies have shown that one of the main reasons students and adults struggle with fractions is because they often overgeneralize properties of whole numbers to fractions (i.e., whole number bias) (NI; ZHOU, 2005). In particular, participants have lower accuracy and slower reaction times when comparing pairs of fractions where the larger fraction has the smaller components (GOMEZ; DARTNELL, 2019). Consistently, when comparing fractions with CC, teachers were slower when comparing incongruent trials than congruent ones. In contrast, when WCC comparing fractions, teachers were slower and less accurate in congruent trials than incongruent trials. Although these findings are in stark contrast to the whole number bias, they could also reflect componential thinking. For example, teachers could have used the smaller denominator-larger-fraction strategy to select the larger fraction, a strategy reported in school children (GOMEZ; DARTNELL, 2019; MILLER; ALISON; BUNGE, 2018). Analysis of distance effects and participants' strategy choices can shed light on this possibility.

Distance effects

Previous studies have posited the lack of understanding of the fractions magnitudes is a source of many students' fraction difficulties (PINTO, 2011; SERRAZINA et al., 2011; HAMDAN; GUNDERSON, 2017; RODRIGUES, 2018), namely, relying on the whole number components of fractions prevents them from considering the numerical value of fractions holistically. Our analyses of distance effects are consistent with the conclusion that math teachers activate holistic fraction magnitudes to solve comparison problems. Specifically, when comparing fractions with and without common components, teachers' reaction times were modulated by the rational distance. Consistent with Obersteiner (2013), we also found places where denominator and numerator distance influence teachers' performance. Given that these distance metrics are correlated (ROSENBERG-LEE, 2021), we used linear mixed effect models to determine the independent contribution of each metric on accuracy and reaction time. In all but one case, it was rational distance that predicted performance after accounting for component distance. Only for the CC-IC did both numerator and rational distance predict reaction times. Overall, these results show that teachers are sensitive to rational number magnitude when comparing fractions, consistent with holistic thinking.

Fraction comparison strategies

We used content analysis to classify participants' explanations of their problem solving into distinct strategies. This methodology is qualitative, and completed by a single researcher, therefore we cannot rule out the possibility of subjectivity in the organizing framework, and other researchers might identify different interpretations (BARDIN, 2011). Based on classifications developed here, we found participants used a wide range of strategies to compare symbolic fractions. In particular, most of the strategies used were the Gap strategy and the numerator and denominator principle strategy. Notably, the very exercise of comparing fractions made some participants re-assess the effectiveness of these strategies, in particular for the Gap strategy. In other words, the description process itself served as an opportunity for the participants to challenge their flawed understanding. That is, this process allowed them to reflect, build new knowledge, and test their hypotheses. Other frequently used strategies were the transformation from fraction to decimal representation (TP: division) and the reference point perspective, most significantly the RPP: ½ distance. Also noteworthy, justifications that reflected componential thinking only comprised 0.7% of teachers' responses, providing no evidence that worse performance on WCC-CO than WCC-IC trials was related to use of the smaller-denominator-larger-fraction strategy. Therefore, based on the JFCT, we can conclude that math teachers participating in this research did not demonstrate a componential view of fractions, corroborating the results revealed by the distance effects analysis of the FCT.

Gap strategy

In the analysis of the explicit justifications in the JFCT, the Gap strategy stood out as the most used strategy. That is participants reported choosing the fraction with the smallest difference between numerator and denominator. This pattern of self-reported strategy use is consistent with teachers' performance on the FCT, namely for the WCC-CO stimuli where the Gap strategy would not always work, teachers performed worse on D_{Gap}-IC and D_{Gap}-N stimuli. Further, considering Gap distance as a continuous metric, we found that accuracy and reaction times were modulated by the Gap distance. Finally, this result remained significant for accuracy, even after account for rational distance. Together, these convergent results from the qualitative and quantitative data question the conclusion that participants are using holistic thinking related only to fraction magnitude. Instead it suggests that subset of participants may be engaged in Gap thinking, which relates to combining the fraction components to compute a simple metric for comparison.

Importantly, the Gap strategy will lead to the correct response in many but not all cases, indicating that teachers are relying on a non-generalizable procedure rather than fully processing the magnitude of fractions. In a study carried out by González-Forte, et al. (2019) with Spanish students from 5th to 10th grade showed that when comparing fractions, younger students were more susceptible to interference from whole number bias than older students. However, older students did not display complete mastery of domain of rational numbers, instead employing mathematically invalid approaches, such as the Gap strategy and the reverse whole number bias, (a component strategy to select fraction with smallest denominator as the largest fraction). The persistence of the Gap strategy found here among mathematics teachers with considerably more training that the 10th grade, suggests that at some point in schooling, students overcome the whole number bias, and even its reverse, but can settle into using use a mathematically invalid strategy indefinitely.

Analyzing the performance of the strategy groups (Gap strategists, Division strategists, and Others) we noticed that for pairs WCC, the strategy that always works led Division strategists to better results, followed by Gap strategists and others, respectively. As for pairs with CC, where the Gap strategy always worked, the Gap strategists obtained better results. This pattern of results suggests that Gap strategists have hit upon an easy to execute and fairly effective strategy. And further some of them many not even be aware that it isn't mathematically valid. In sum, the findings of the rational distance effect from the FCT suggest holistic process, that is participants hard rapid access to the rational magnitude of each fraction in the pair. Yet, the fact that the majority of participants used Gap and Division strategies, means we cannot affirm the holistic view, since the most strategies used by teachers were either erroneous or based on an arithmetic procedure.

Educational implications

Mixing different fields of knowledge (mathematical education and cognitive psychology) with different methodologies (qualitative and quantitative analyses) afforded us insights into our research questions that pursuing only one path would not have allowed. We found that participants use a variety of strategies to compare fractions. However, there is a set of knowledge that is not observed in textbooks or in teaching practice. We noticed the consistent use of the Gap strategy, which is not mathematically valid, but on the other hand, there is valid and important knowledge that could be used in the teaching and learning process of fractions, especially to estimate their magnitude. Therefore, a reorganization of the teacher's knowledge is shown to be an important step that needs to be carried out.

This novel finding that many skilled teachers employ the mathematical invalid Gap strategy leads to new questions regarding teachers' educational practice, especially given the relationship between teacher knowledge and student learning (HILL; ROWAN; BALL, 2005; DEPAEPE et al, 2015). For example, were teachers explicitly taught this strategy, or did they discover it themselves. Also, do teachers promote these strategies while teaching? Further, the intriguing observation that some teachers noticed that their approach was incorrect while solving D_{Gap}-N problems, raises an interesting possibility: would confronting teachers with the limits of this strategy provide a means of illuminating its shortcomings and altering their practice? More broadly, a promising teaching approach might make it explicit that people (even specialist math teachers) are influenced by their intuitions while reasoning mathematically, that these intuitions can be misleading, and that conscious awareness and control are necessary full mathematical understanding. The fact that some teachers noticed their misunderstanding is an intriguing phenomenon. In a study conducted by Rittle-Johnson (2006) with children aged 8 to 11 years, self-explanation helped them learn and remember a correct procedure, as well as promoting better transfer, relative to students who were not required to self-explain. As similar effect may be at work here, where self-explanation served as an evaluation process, allowing participants to refute their convictions, reflect, and then test hypotheses to build new knowledge. Future studies could examine the effects of confront the inadequacies of the Gap strategy in prompting change in strategy use, for both students and teachers.

More broadly, we can considered how the Gap strategy use fits within the larger class of fraction difficulties and misunderstandings. Siegler and Lortie-Forgues (2017) identify two main classes of difficulties underlying the misunderstanding of rational number arithmetic: culturally contingent and inherent. Culturally contingent sources of difficulty are those that vary across cultures, such as a teacher's understanding of rational numbers. They lead to poorer learning among students in some places than in others. The inherent sources of difficulty are those imposed by the number system itself, and are present for all students. Extending this distinction from fraction arithmetic understanding of fractions themselves, the overuse of the Gap strategy may be an example of a culturally contingent source of difficulties for fraction comparison, especially if it is explicitly taught. Future studies could examine whether teachers in other countries also rely on the Gap strategy and are unaware of its limitations.

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