CEMeR
Caminhos da Educação Matemática em Revista
2023 • Ano X • V 13 • n. 1 • p. 76-92
ISSN 1983-7399

ARTIGO ORIGINAL/ ORIGINAL ARTICLE

Arthur Belford Powell, Rutgers University, USA (1)

Two perspectives of fraction knowledge: characterization, origins, and implications

Duas perspectivas do conhecimento de frações: caracterização, origens e implicações

ABSTRACT
Different views of fraction knowledge emerge from distinct origins and imply varying consequences for mathematics teaching and learning. Underscoring variant theoretical positions of fraction knowledge, we probe views of how young learners appropriate natural and fractional numbers. We advance a perspective that awareness of fractional numbers results from sensing a quantitative relation between the magnitude of two quantities, using the psychological mechanisms of stressing and ignoring. Afterward, focused on the quantitative relations, we present a mathematical analysis of the magnitude of two quantities. Building on the theoretical views of fractions, we outline the mathematical and cognitive attributes of two models of fraction knowledge—partitioning and measuring—describing their historical and philosophical-technological origins. Finally, we describe the cognitive consequences of the partitioning perspective and suggest areas for future research to investigate further the measuring perspective.

Keywords: Measuring Perspective, Partitioning Perspective, Fraction Knowledge.

RESUMO
Diferentes visões do conhecimento fracionário emergem de origens distintas e implicam consequências variadas para o ensino e a aprendizagem da matemática. Ressaltando posições teóricas variantes do conhecimento fracionário, investigamos visões de como os jovens aprendizes se apropriam de números naturais e fracionários. Avançamos uma perspectiva de que a consciência de números fracionários resulta da percepção de uma relação quantitativa entre a magnitude de duas quantidades, usando os mecanismos psicológicos de enfatizar e ignorar. Em seguida, com foco nas relações quantitativas, apresentamos uma análise matemática da magnitude de duas grandezas. Com base nas visões teóricas das frações, delineamos os atributos matemáticos e cognitivos de dois modelos de conhecimento fracionário – particionamento e medição – descrevendo suas origens históricas e filosófico-tecnológicas. Finalmente, descrevemos as consequências cognitivas da perspectiva de particionamento e sugerimos áreas para pesquisas futuras para investigar mais a perspectiva de medição.

Palavras-chave: Perspectiva de medição, perspectiva de particionamento, conhecimento de frações

(1) Ph.D. in Mathematics Education - Rutgers University, USA. Member of the Ph.D. Program in Global Urban Studies – Rutgers University, USA. Ph.D in Education and Learning Sciences - Rutgers University, USA.

Correspondência:
powellab@newark.rutgers.edu (1)

Recebido em 21/03/2022
Aprovado em 31/08/2022

INTRODUCTION

Implementing contrasting epistemological perspectives of mathematical content invariable produces different outcomes. How to support learners' meaningful engagement with fraction knowledge has been a significant challenge for mathematics educators. Teachers struggle to help students from their early years of schooling conceptualize and operate on fractions (Zhou et al., 2006). Students who experience difficulties with fractions incorrectly order and operate with fractions (Maher & Yankelewitz, 2017). Furthermore, they underachieve in mathematics and have unsuccessful experiences learning higher mathematical topics such as algebra (Fuchs et al., 2017; Siegler et al., 2012; Torbeyns et al., 2015).

Recognizing the fundamental importance of fraction knowledge for success in studying mathematics beyond arithmetic, in this Article, we aim to highlight two contrasting theoretical positions of fraction knowledge to understand their cognitive and mathematical attributes and the challenges they pose. To appreciate the two positions, we first examine how young learners appropriate natural and fractional numbers. From that examination, we advance a perspective that the knowledge of fractional numbers follows from becoming aware and making sense of a quantitative relation between the magnitude of two quantities, using the psychological mechanisms of stressing and ignoring. Afterward, focused on the quantitative relations, we present a mathematical analysis of the magnitude of two quantities. Building on the theoretical views of fractions, we outline the mathematical and cognitive attributes of two models of fraction knowledge—partitioning and measuring—describing their historical and philosophical-technological origins. Then, we describe cognitive consequences of the partitioning perspective. Finally, we suggest areas for future research aimed at investigating further the measuring perspective.

MODELS OF FRACTIONS

Before and especially during their first seven or eight years of schooling, children learn natural, rational, and then integral numbers along with the arithmetic of those numbers. Concerning natural and fractional numbers, how young learners appropriate linguistic and semiotic knowledge of them has been particularly interesting to researchers in fields ranging from anthropology and linguistics to mathematics education and neurocognitive psychology (see, for example, Blair et al., 2012; Chan et al., 2022; Chrisomalis, 2015; Davydov & Tsvetkovich, 1991; Mendes et al., 2004; Rosenberg-Lee, 2021; Steffe, 2001). Following this interest, we reflect on how learners appropriate natural numbers and fractions through abstraction. We understand appropriation of knowledge as a constructive process that occurs from social and cultural sources through goal-directed and tool-mediated actions. Those actions lead learners to construct and internalize their version of those social and cultural ideas.

From this perspective of appropriation, Davydov and Tsvetkovich (1991) posit acquiring natural numbers and fractions involves abstracting at two different cognitive levels. First, the idea of natural numbers entails abstracting from the qualitative attributes of objects to their quantity and magnitude. Here learners consider a quantity as a countable or measurable quality or attribute of an object such as its individuality, length, area, volume, or weight. From a collection of identical or non-identical things, we separately contemplate their "objectness" and determine how many there are, that is, their magnitude. From the various qualitative attributes of objects, learners abstract quantity and magnitude at this level and, thereby, appropriate counting or natural numbers—1, 2, 3, and so on.

Second, supported by the previous appropriation process, Davydov and Tsvetkovich (1991) theorize that grasping the idea of fractions engages a higher mental functioning, an abstraction of an abstraction. It involves conceiving a "quantitative relationship…between objects, devoid of all qualities" (p. 87). At this higher cognitive level, the abstraction concerns a quantitative relation between abstracted magnitudes appropriated at the previous level. Specifically, learners abstract a quantitative relation between the magnitude of two quantities.

The abstracting act at each of the two levels can be described differently and address quantifying discrete as well as non-discrete entities. Gattegno (1970d, 1973) posits abstraction's mechanism as the twin mental functionings he terms stressing and ignoring. Considering those mental functionings, learners can achieve the first abstraction level by ignoring the quality of individual things in a collection or continuous entity and stressing their quantitative "objectness" to arrive at a collection or entity's magnitude. At that moment, by having counted, they can answer the question, "how many?" For the magnitude of a continuous entity, learners may have to count the iteration of a unit until it reaches, for instance, the entity's length, area, or volume. To arrive at the second abstraction level, learners must ignore all attributes other than the magnitude of two quantities and stress a quantitative relation between the magnitudes. Finally, they can respond to a comparative question, "how much?" one quantity is of the other.

MAGNITUDES' QUANTITATIVE RELATIONS

Quantitative relations between the magnitude of two quantities can be additive or multiplicative. In the additive case, the comparative magnitude of two different quantities— p and q —yields one of two relations: depending on whether $p > q$ or $p < q$, either $p - q = k$ or $q - p = k$, respectively, where p, q, and k are natural numbers[1].

In the multiplicative case, the comparative relation of two quantities p to q means for some m and n, $n \times p = m \times q$. Therefore, $n/m \times p = q$ and $m/n \times q = p$. Consequently, a fraction represents a "quotient of two quantities of the same dimension, expressed in the same unit" (Vergnaud, 1983, p. 162). In other words, a fraction describes a multiplicative comparison between two commensurate quantities of the same kind. That idea is encapsulated in its symbolic form, p/q, composed of two natural numbers. However, for some learners, a fraction's bipartite form and infinite equivalent expressions, $n \times p/n \times q$, may belie the idea that its form and equivalent expressions represent a single, unique magnitude.

Also significant is the material or conceptual relationship of the two quantities to each other. In a specific instance, that material or conceptual relationship has one of two characteristics. Describing the characteristics, Vergnaud (1983) notes, "either one quantity is part of the other (inclusive case) or there is no obvious inclusion relationship (exclusive)" (p. 162).

[1] Natural numbers are defined as the set of non-negative integers $\{0, 1, 2, 3, \ldots\}$ or, following Peano's axioms, the set of positive integers $\{1, 2, 3, \ldots\}$. Here, we consider the latter set when referring to the natural numbers.

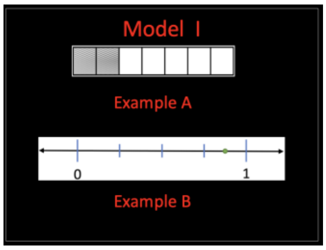
The inclusive case is a whole, q, and a part, p, of it (p out of q), such as this set model statement: Aaliyah ate two-thirds of the cookies. For an area model, one compares a number, p, of pizza slices to the number of slices, q, in the whole pizza.

In contrast, the exclusive case involves comparing two distinct quantities, p and q, that have no inclusion relationship (p to q). For example, the volume of Karma's luggage is three-fifths of Samir's luggage. The second volume, q, is the unit to which the first, p, is compared. The quantities are of the same kind (volume) and compare multiplicatively. As a further instance of the exclusive case, Davydov and Tsvetkovich (1991) present this situation: A student compares two distinct objects, sharing length as a common quantifiable attribute or quantity, a ruler to a table's side. To compare the objects, the student assigns one of the quantities to equal one or the unit and measures. For example, if n iterations of the student's ruler equal the table's side, and she considers the table the unit, the ruler is one-nth or $1/n$ of the table.

COGNITIVE AND MATHEMATICAL FEATURES OF MODELS I AND II

There are subtle yet significant cognitive and mathematical differences between the two cases. We will enumerate some differences but, first, illustrate the two cases. The inclusive and exclusive cases are depicted respectively in the models in Figures 1 and 2. In Figure 1, Model I includes two examples of the inclusive case. Model I's graphics are like familiar textbook illustrations of an area and a number-line presentation used to introduce fractions. The area presentation in Example A shows a rectangle divided into seven smaller rectangles with two shaded, showing 2/7 of the original area. Example B highlights a number-line point in green midway between two tick marks. As the three tick marks equipartition the length between 0 and 1, they represent the fractions $1/4, 2/4$, and $3/4$; consequently, the green point is $7/8$ of the distance away from zero.

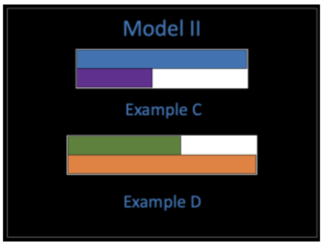
Figure 1 – Model I, two examples of inclusive models for illustrating fractions



Source: Author's file.

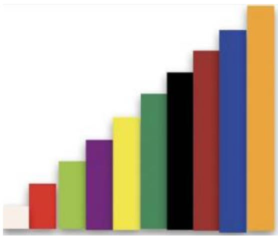
Figure 2 contains two examples of the exclusive case for representing fractions. Model II's graphic examples employ a two-dimensional representation of Cuisenaire rods. A set of Cuisenaire rods constitutes ten different-sized rectangular parallelepipeds or cuboids; the length of each is a multiple of the smallest—a one-centimeter cube. Each size is colored uniquely from shortest to longest: white, red, lime, purple, yellow, green, ebony, chocolate, blue, and orange. In Model II, the length of the blue, purple, green, and orange rods are respectively equal to 9, 4, 6, and 10 centimeters (see Figure 3). Example C represents the fraction 9/4 or 4/9, depending on which rod is considered the unit of measure. Similarly, again relative to which rod is designated the measuring unit, Example D shows the fraction 6/10, 3/5, 10/6, or 5/3.

Figure 2 – Model II, two examples of exclusive models for illustrating fractions



Source: Author's file.

Figure 3 – Cuisenaire rods



Source: Author's file.

We can now enumerate cognitive and mathematical differences between the two representational models of fractions. Model I reveals five cognitive and mathematical attributes. First, whether an area, number line, or set, the representation is an equipartition, discretized quantity. Second, as such, it invites counting the individual partitions or discrete objects. Third, from the learner's perspective, the whole or unit is implicit and predetermined. Fourth, because of the previous feature, what is the unit fraction, where the numerator equals 1, is also predetermined. Finally, a fraction is a quantity denoting counts of discrete entities, the number of highlighted partitions—shaded area, tick marks, or individual objects—to the total number of partitions. Furthermore, aside from those five attributes, a salient feature of Model I is that an essential mathematical idea—a unit—is treated implicitly and, therefore, can evade learners' and teachers' careful consideration (Ciosek & Samborska, 2016; Powell et al., 2022).

Contrastingly, Model II's cognitive and mathematical attributes engage learners explicitly in making decisions about what the unit is. Like the previous model, Model II comprises five features. First, the representation contains two objects with a common quantity, such as length (or even area, volume, or weight). Second, the quantity of one object is considered the unit of measure and used to compare the other object's quantity multiplicatively. Third, comparing the two objects multiplicatively necessitates either estimating or measuring. Fourth, measuring requires deciding what will be the unit fraction and iterating it to determine the measure of the object that is not the unit of measurement. Finally, a fraction expresses a relation, a multiplicative comparison between two quantities of the same kind.

Figure 4 summarizes the contrasting cognitive and mathematical features of the two fundamental cases of fraction knowledge. The first represented the inclusive case, which we call the partitioning perspective. The second exemplified the exclusive case, and we label it the measuring perspective. Unlike the partitioning perspective, functioning within the measuring perspective of fraction knowledge requires deciding explicitly which common quantity of two objects to focus on and which object's quantity to take as the unit of measure.

Figure 4 – Cognitive and mathematical features of two perspectives of fraction knowledge

Cognitive and mathematical attributes of fraction-knowledge perspectives

Partitioning	Measuring
1. One equipartitioned, discretized quantity or whole	1. Two distinct continuous quantities
2. A whole or unit is implicit and predetermined or "preordained"	2. One quantity is the explicit unit
3. Invites counting	3. Necessitates estimating or measuring
4. Unit fraction is also predetermined	4. Unit fraction is determined
5. A fraction denotes counts of discrete entities: number of highlighted parts to the total number of parts.	5. A fraction expresses a multiplicative relation: one quantity measured by another quantity of the same kind.

Source: Author's file.

ORIGINS OF THE TWO PERSPECTIVES OF FRACTION KNOWLEDGE

Measuring Perspective

The two perspectives of fraction knowledge—partitioning and measuring—have not only fundamentally different cognitive and mathematical features but also disparate origins. For example, the measuring perspective originates on the African continent in ancient Egypt among property surveyors. In contrast, the partitioning perspective arose in reaction to a philosophical movement among professional mathematicians and a technological advance.

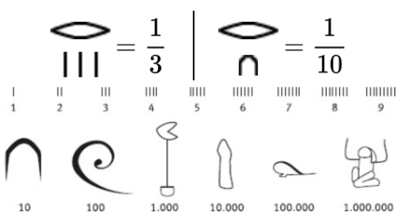
The measuring perspective derives from ancient African social-cultural practices. More than four millennia ago, in Mesopotamian and Egyptian cultures, along the Tigris, Euphrates, and Nile rivers, with the birth of agriculture, the material conditions introduced the need to measure quantities of land, crops, seeds, and so forth and to record the measures (Clawson, 1994/2003; Struik, 1948/1967). More than 3400 years ago, to measure land distances, ancient surveyors stretched knotted ropes in which the length between knots represented a unit of measure (see Figure 5). When the land distance was not an exact multiple of the unit, the need for fractional lengths emerged. The lengths were a relation representing a multiplicative comparison between two quantities: (1) a dimension of land, a distance d and (2) a unit of measure u, the uniform distance between consecutive knots. From the social-cultural practice of surveying arose simultaneously fractional numbers and geometry (Aleksandrov, 1963; Caraça, 1951; Roque, 2012). Later, ancient Greeks discovered that such ratios of lengths were not always measurable by a standard unit or commensurable (Struik, 1948/1967), leading to the discovery of irrational numbers. Corresponding to this cultural-historical perspective, ontologically, we define a fraction as *a multiplicative comparison between two commensurable quantities of the same kind*.

Figure 5 – A wall painting[2] from the Tomb of Menna (TT69) of the Theban necropolis in Luxor's West Bank depicting surveyors measuring land with a knotted rope. Menna, an ancient Egyptian official, was a scribe and an overseer of fields belonging to the pharaoh Amenhotep III and the temple of Amun-Re (Hartwig, 2021)

Source: Author's file.

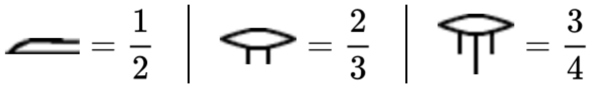
More than 4,000 years ago, ancient Egyptians developed written versions of fractions. Specifically, to represent fractions, they invented two forms (Roque, 2012):

1. Unit fractions are designated with an oval symbol placed above the number n to represent what today we denote with $\frac{1}{n}$.



[2] By Charles Wilkinson - This file was donated to Wikimedia Commons as part of a project by the Metropolitan Museum of Art. See the Image and Data Resources Open Access Policy, CC0, https://commons.wikimedia.org/w/index.php?curid=89562438

2. Special fractions, including 1/2, 2/3, and 3/4, were represented with specific hieroglyphic signs.



Since the Egyptian invention of written fractions emerged around 1800 BCE, their notion and use of fractions have evolved. However, it was only in the XVII century that mathematicians accepted fractions as legitimate mathematical objects like natural numbers as they permitted equations of the form $ax = b$ to have solutions, $x = b/a$, without restriction when $a \neq 0$. Furthermore, they provided meaning to the division of any pair of natural numbers, resulting in a closed domain, an algebraic field (Courant & Robbins, 1941/1996). Interestingly, the Egyptian notion of a fraction as an object representing a multiplicative comparison between two quantities persisted until, historically speaking, recently. Then, as we shall now discuss, the idea of fractional numbers emerging from measuring, spurred by material conditions, becomes supplanted by fractions rooted in a partitioning perspective.

Partitioning Perspective

The conditions for the current, dominant perspective of fraction knowledge—partitioning—arose in the first half of the 20th century with a part/whole interpretation of fractions. The movement from the historical origins of fractions in the act of measuring two quantities, one considered the unit measurement, has two confluent roots, one philosophical and the other technological (Davydov & Tsvetkovich, 1991; Escolano Vizcarra, 2007; Schmittau, 2003). One impetus was in reaction to philosophical debates among mathematicians. Specifically, at the beginning of the 20th century, the philosophy of formalism, championed by David Hilbert (1862-1943), proposed a program to axiomatize mathematics completely and consistently (Snapper, 1979). At the time, formalists maintained that the objects of mathematical thinking are the mathematical symbols themselves and not any meanings attributed to them (Simons, 2009). On this point, the following statement, attributed to Hilbert, roughly summarizes the core position of the formalist project: “Mathematics is a game played according to certain simple rules with meaningless marks on paper” (Rose, 1998). This philosophical belief about the nature of mathematics permeated mathematical education.

A consequence of formalism is how rational numbers are defined, especially fractions (Schmittau, 2003). A formalist definition of rational numbers: Rational numbers represented as common fractions are bipartite symbols that express quotients or ratios of two whole numbers, a/b, such that a and b are natural numbers and $b \neq 0$. In the expression, a/b, a is called the dividend or numerator, and b is the divisor or denominator.

Within mathematics education circles, educators understood that the formalist definition of fractions was too abstract for students. Consequently, they derived a pictorial version of the formalist symbolic notation designed to aid students' understanding of the numerical quotient conception of fractions. As Davydov and Tsvetkovich (1991) detail,

> psychological factors also had to be considered. It was impossible to give fifth graders, to say nothing of younger students, a purely symbolic outline of the principle of division which produces fractions. A *visual* correlate was needed. This role was given to the so-called division of things, their subdivision into

> parts, which in the course of instruction could be linked relatively easily with terms characteristic in defining common fractions. Thus, the "visual conception of the fraction"...arose completely naturally. By rather intricate means, the original source of the formation of fractions in the history of teaching mathematics-—measurement—was replaced in didactic goals through the substitution of the so-called "division of things" for the division of numbers. Such a conception was used widely not only in the upper grades, but even in the primary grades-so much so that it was natural to connect the isolation of "portions" with the child's experience in dividing objects such as apples or watermelons. (pp. 100, original emphasis)

The partitioning perspective's division of sets and areas into equal parts, the part/whole interpretation, provides a "visual correlate" to the formalist definition of a fraction as the quotient, a/b, of two integers a and b, where b is nonzero. This inclusive perspective substituted the historical, exclusive view of fractions. The inclusive view of fraction knowledge has another source related to the instructional goal of supporting learners' understanding.

Concurrent with the influence of formalism, a second source for transitioning the teaching of fractions from a measuring perspective to a part/whole interpretation was a technological innovation for providing printed visual aid in textbooks. By the 1930s, improvements in printing technology enabled book publishers to present two-dimensional graphic images in pages of mathematics textbooks (Escolano Vizcarra, 2007).

Before that innovation, textbooks explicitly asked students to compare magnitudes of objects. Instead, the new textbooks presented a single geometric figure with an area subdivided into equal parts, and students had now to relate the nonsymbolic graphic image to symbolic fractional notation. Escolano Vizcarra (2007) explains: "The exercise proposed by the textbook does not refer to any measurement process since neither the magnitude nor the unit of measurement is made explicit" (pp. 81, my translation, added emphasis). It is crucial to underscore that the absence of magnitude (a quantity) and unit of measurement (a second quantity) implies a complete movement from an exclusive to an inclusive perspective, and the idea of unit shifts from being explicit to existing only implicitly.

Escolano Vizcarra (2007) continues:

> The *part-whole relationship* appears because of a gradual process of abandoning the meaning of measurement with real objects... Consequently, it is a didactic resource that arises to avoid measurement activities with tangible objects, possibly because the measurement processes in the classroom generate difficulties such as material management, control of the diversity of results obtained or the appearance of interferences with the teaching of the Decimal Metric System. (pp. 81, my translation, original emphasis).

Thus, it appears that two confluent events conspired to supplant the historical measuring perspective of fraction knowledge with one that emphasizes equipartitioning of geometric areas to ground and introduce fractional numbers to young learners. The first event was the technology for rendering 2-D images on a printed page and second the consequent instructional and material neatness of a visual part/whole interpretation of fractions both conspired.

COGNITIVE CONSEQUENCES OF THE PARTITIONING PERSPECTIVE

Now that the partitioning perspective is dominant, it is essential to review its cognitive consequences for learners. Three implications of the perspective are paramount: natural number ideas, numerical magnitude, and symbolic notation. Fractions, one of rational numbers' three representations, conceptually present specific cognitive challenges related to their symbolic form. As a mathematical construct, a fraction is a sign with a bipartite structure composed of two natural numbers. Nevertheless, a fraction signifies a single number but derives its notational and conceptual meaning from its pair of natural numbers.

Notwithstanding their natural number components (numerator and denominator), fractions have properties that differ from natural numbers, such as larger numerals signal larger numerical magnitudes. Moreover, like the numbers that compose it, a fraction represents a magnitude and is locatable on a number line. The fraction's magnitude results from a particular relation between its components. Specifically, the relation is a measure, a multiplicative comparison between the two natural numbers that compose the fraction, where the denominator is the unit of measurement. Those characteristics of fractions' symbolic notation and magnitude and how to address them epistemologically and pedagogically are cognitively challenging and mathematically essential.

According to Kieren (1976, 1980), the fractional form of a rational number has five sub-constructs or interpretations: part/whole, quotient, measure, ratio, and operator. Nevertheless, in the early years of education, the part/whole interpretation is dominant. With this interpretation, students build the following three aspects of fraction knowledge:

1. the origin of fractions is in the division of things into equal parts;
2. a fraction is a description of two related discrete attributes of an object: (a) a collection or a discretized whole and (b) a certain number of parts of it; and
3. counting is the procedure of naming a part of the division of a thing in fractional terms.

Those aspects of the part/whole or partitioning perspective present epistemological challenges. Since the view engages counting and an additive design, students use natural number ideas (concepts, properties, and procedures) to make sense of fractional numbers and operations. Researchers refer to such use as the "whole number bias" or "the natural number bias" (Ni & Zhou, 2005; Thompson et al., 2021; Van Dooren et al., 2015; Van Hoof et al., 2015). This bias manifests in Mack's (1990) observation that 6th-grade students (11 years old) tend to state that 1/8 is greater than 1/6, since 8 is larger than 6. Another evidence of the natural number bias occurs when a student describes adding fractions as follows: "Well, you cross. You add the top numbers together and the bottom numbers together," and consequently, the sum of 1/2+1/3 becomes 2/5 (Mack, 1990, p. 23). Furthermore, in this volume, Toledo et al. (2022) report on mathematics teachers who use a non-generalizable strategy for comparing fraction magnitude based on a property of natural numbers and

The natural number bias evidences other challenges of learners' fraction sense[3]. For

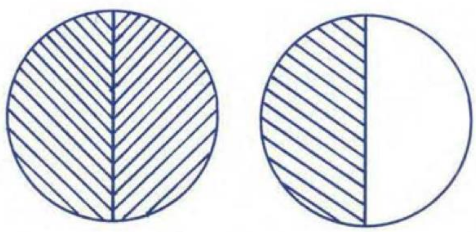
[3] See Powell, A. B., & Ali, K. V. (2018). Design research in mathematics education: Investigating a measuring approach to fraction sense. In J. F. Custódio, D. A. da Costa, C. R. Flores, & R. C. Grando (Eds.), *Programa de Pós-Graduação em Educação Científica e Tecnológica (PPGECT): Contribuições para pesquisa e ensino [The Graduate Program in Science and Technology Education (PPGECT): Contributions for research and teaching]* (pp.

example, they have difficulty understanding that multiplying two positive fractions can produce a result smaller than either of two original fractional factors, while division may produce a quotient greater than the operands, depending on the magnitude of the multiplier and divisor (Siegler et al., 2013; Vamvakoussi & Vosniadou, 2004). Similarly, learners find it difficult to understand that, unlike natural numbers, fractional numbers are a dense subset of real numbers with an infinite collection of fractions between any two fractions (Maher & Yankelewitz, 2017; Ni & Zhou, 2005; Siegler, 2016; Vamvakoussi & Vosniadou, 2004, 2010). Thus, learners also fail to recognize that, different than natural numbers, fractions have no distinct successor, the logical opposite of fractions' density property.

Further evidence of inadequately constructed fraction sense concerns the basic idea of units. When operating on fractions, learners fail to consider their magnitudes relative to a unit. For instance, in 1987, a nationally representative sample of more than 20,000 U.S. 13- and 14-year-olds (8^{th} graders) was asked to estimate—not to calculate—the closest whole number for the sum of $\frac{12}{13}+\frac{7}{8}$ from among five choices: 1, 2, 19, 21, or "I don't know." Most students chose "19" or "21" with "19" the most common answer, suggesting that they added the numerator or denominator and not the given fractions. Only 24% of the students chose the correct answer, "2," indicating that they likely estimated each fraction's magnitude to equal 1 (Carpenter et al., 1980). More than three decades later, Lortie-Forgues, Tian, and Siegler (2015) administered the same question to a sample of 48 8th-grade (13- and 14-year-olds) algebra students who attended, as the authors describe, "a suburban middle school in a fairly affluent" community. Of their sample, only "27% of the 8^{th} graders identified '2' as the best estimate of 12/13+7/8" (p. 202). To add the two fractions, most students apparently did not sense the reasonableness of attending to the fractions' magnitudes and their relation to a unit.

Another symptom of instructional inattention to the unit concept concerns naming the fractional part of a pictorial model. For example, in Figure 6, Gattegno and Hoffman (1976) question whether students should be faulted for thinking that the shaded regions represent 3/4, 3/2, or even 3/1 without the task stating what the unit is. Depending the unit, any one of the responses can be valid. However, instructional materials typically present students with one pizza or a chocolate bar as the unit. Consequently, students have difficulty imagining the unit as two pizzas or four sections of an eight-sectioned chocolate bar. Presented in different instantiations, students need to experience and reflect on the *unit* idea, a fundamental concept for constructing fraction knowledge (Behr et al., 1997; Campos & Rodrigues, 2007; Gattegno & Hoffman, 1976; Lamon, 1996). Equally as essential is the unit fraction concept.

Figure 6 – What fraction is represented by the shaded parts?



Source: Gattegno and Hoffman, 1976, p. IA5

221-242). Livraria da Física. for a discussion of fraction sense.

From a dialectical materialist perspective, everything is in a continual process of becoming and ceasing to be, nothing is permanent, and everything changes and is eventually superseded. It may also be the case with perspectives of fraction knowledge. However, given the documented challenges of a partitioning perspective for understanding fractions, further research is warranted to understand how an initial introduction to fractions from a measuring perspective might mitigate those challenges and provide a firmer cognitive basis for later grasping the other five interpretations of fractions within the partitioning perspective.

CLOSING REMARKS

We want to outline research areas to strengthen the understanding of cognitive and instructional efficacy of a measuring perspective of fraction knowledge. The measuring perspective has been theoretically and pedagogically investigated by Brousseau et al. (2004, 2008) and Davydov and others who have adapted the Elkonin-Davydov curriculum (Davydov & Tsvetkovich, 1991; Dougherty & Venenciano, 2007; Morris, 2000; Schmittau & Morris, 2004). Morris (2000) investigated the adoption of the Elkonin-Davydov approach to fraction learning with a group of fourth-graders who were learning about fractions as numbers. The contributions of Morris (Morris, 2000), Schmittau and Morris (2004), Brousseau et al. (2004, 2008), Davydov and Tsvetkovich (1991), and Dougherty and Venenciano (2007) to understanding affordances of a measuring perspective for fraction knowledge are essential. However, they raise questions about how this perspective can help learners conceptualize and interpret fractions represented in continuous and discrete models. To work with these models, learners must manage measuring, partitioning, and unitizing across various representational models.

Cuisenaire and Gattegno (Cuisenaire & Gattegno, 1954; Gattegno, 1970a, 1970b, 1970c) do not stipulate numerical values (number names) for each Cuisenaire rod. Instead, they code rod lengths with letters corresponding to their color names and subsequently encode them with numerical values corresponding to their measure with one rod considered as the unit (of measurement). Their approach not only corresponds to the historical measuring perspective but also to Davydov's (1962) insight about the importance of measuring as an initial introduction to numbers:

> Such introduction of whole numbers greatly facilitates the subsequent mastering of fractions—both simple and decimal—since the child understands from the very outset, first that abstract number as a relationship, and, second, the value being measured as a homogeneous object that may be measured with any degree of precision. (p. 35)

Without using Cuisenaire rods, a study that employed a measuring perspective with prospective teachers was conducted by Bobos and Sierpinska (2017). They engaged future teachers with Sierpinska's (2005) model of theoretical thinking to reason about quantities and conceptualize fractions from Davydov and Tsvetkovich's (1991) perspective, where numbers represent "measures of how-much-ness of one quality relative to another quality of the same kind" (Bobos & Sierpinska, 2017, p. 205, p. 205). The qualities included ideas like two-ness of sets of objects or sweetness or salinity of liquids. Their results show the "connection between the material [measurement situations] and the formal conceptions of fractions remains difficult to achieve" (p. 234). To this point, their implementation did not involve using manipulatives applicable to varying situations of visualizing fractions and operations. Their prospective teachers had to draw models without access to such multivalent manipulatives. They may have been more preoccupied with selecting and drawing appropriate visual representations of fractions and fraction operations than underlining conceptual ideas. Consistent use of multivalent manipulatives may reduce learners' burden to accurately depict their representations and provide learners with more mental space to entertain the relevant fractional ideas appropriately. We suggest studies that address this issue by investigating one set of manipulatives that allow for representing, comparing, and operating on fractions.

Growing body of evidence demonstrates that humans have a ratio processing system (RPS), a neurocognitive architecture that supports comparing the magnitudes of two non-symbolic ratios (Geary et al., 2015; McCrink & Wynn, 2007; Meert et al., 2013). There is evidence of important links between non-symbolic, continuous models of ratios and fractions (Matthews et al., 2016; Bhatia et al., 2020; Kalra et al., 2020). How students use their reasoning about non-symbolic continuous ratios to think about symbolic fractions is an area requiring investigation.

In this volume, Abreu-Mendoza and Rosenberg-Lee (2022) review studies building on previous evidence suggesting young children work well with nonsymbolic representations of proportions, noting that those processing skills are also positively related to fraction ability in older children and adults. Those findings spurred the studies of instructional interventions designed to leverage nonsymbolic skills and enhance symbolic fraction understanding. Their review concludes that studies provide evidence, small-to-large effect sizes, that pairing nonsymbolic and symbolic representations of fractions on number lines improved fraction skills of low and typically-achieving students. They suggest fostering nonsymbolic skills may be important in addressing challenges of fractions knowledge.

The number line is an inclusive representation of fractions. Complementing the studies Abreu-Mendoza and Rosenberg-Lee (2022) reviewed, research is needed to examine how pairing an exclusive case of nonsymbolic fractions, using appropriate manipulative materials such as Cuisenaire rods, and symbolic representations of fractions shapes learners' understanding of fraction magnitude.

Finally, other research needed concerns teacher enhancing their knowledge from a measuring perspective. In wo different studies, Alqahtani and Powell (2018) and Alqahtani et al. (2022) report changes in pre-service teachers' awareness of unit and conceptual understanding of fractions. For practicing teachers' professional learning, in this volume, Souza and Powell (2022) analyze a 12-week Lesson Study project involving 11 Brazilian teachers learning and designing to teach fractions from a measuring perspective. Further research is needed to understand how teachers revise their teaching practice as their implement measuring perspective, using appropriate manipulative materials.

REFERENCES

Abreu-Mendoza, R. A., & Rosenberg-Lee, M. (2022). Measuring fences and sharing pizzas: Current advances in nonsymbolic fraction interventions. *Caminhos da Educação Matemática em Revista*.

Aleksandrov, A. D. (1963). A general view of mathematics (S. H. Gould & T. Bartha, Trans.). In A. D. Aleksandrov, A. N. Kolmogorov, & M. A. Lavrent'ev (Eds.), *Mathematics: Its content, methods, and meaning* (Vol. 1, pp. 1-

64). Massachusetts Institute of Technology.
Alqahtani, M. M., & Powell, A. B. (2018). Investigating how a measurement perspective changes pre-service teachers' interpretations of fractions. In T. E. Hodges, G. J. Roy, & A. M. Tyminski (Eds.), *Proceedings of the 40th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 484-487). University of South Carolina & Clemson University.
Alqahtani, M. M., Powell, A. B., Webster, V., & Tirnovan, D. (2022). How a measuring perspective influences pre-service teachers' reasoning about fractions with discrete and continuous models. *International Electronic Journal of Elementary Education, 14*(3), 441-458.
Behr, M. J., Khoury, H. A., Harel, G., Post, T., & Lesh, R. (1997). Conceptual Units Analysis of Preservice Elementary School Teachers' Strategies on a Rational-Number-as-Operator Task. *Journal for Research in Mathematics Education, 28*(1), 48-69. https://doi.org/10.2307/749663
Blair, K., Rosenberg-Lee, M., Tsang, J., Schwartz, D., & Menon, V. (2012, 2012-February-10). Beyond Natural Numbers: Negative Number Representation in Parietal Cortex [Original Research]. *Frontiers in Human Neuroscience, 6*. https://doi.org/10.3389/fnhum.2012.00007
Bobos, G., & Sierpinska, A. (2017). Measurement approach to teaching fractions: A design experiment in a pre-service course for elementary teachers. *International Journal for Mathematics Teaching and Learning, 18*(2), 203-139.
Brousseau, G., Brousseau, N., & Warfield, V. (2004, //). Rationals and decimals as required in the school curriculum: Part 1: Rationals as measurement. *The Journal of Mathematical Behavior, 23*(1), 1-20. https://doi.org/https://doi.org/10.1016/j.jmathb.2003.12.001
Brousseau, G., Brousseau, N., & Warfield, V. (2008, //). Rationals and decimals as required in the school curriculum: Part 3. Rationals and decimals as linear functions. *The Journal of Mathematical Behavior, 27*(3), 153-176. https://doi.org/https://doi.org/10.1016/j.jmathb.2008.07.006
Campos, T. M. M., & Rodrigues, W. R. (2007). A idéia de unidade na construção do conceito do número racional. *REVEMAT - Revista Eletrônica de Educação Matemática, 2*(4), 68-93.
Caraça, B. d. J. (1951). *Conceitos fundamentais da Matemática*. Tipografia Matemática.
Carpenter, T. P., Corbitt, M. K., Kepper, H. S., Lindquist, M. M., & Reys, R. E. (1980). Results of the second NAEP mathematics assessment: Secondary school. *Mathematics Teacher, 73*(5), 329-338.
Chan, J. Y.-C., Sera, M. D., & Mazzocco, M. M. M. (2022). Relational language influences young children's number relation skills. *Child Development*. https://doi.org/doi.org/10.1111/cdev.1373
Chrisomalis, S. (2015). What's so improper about fractions? Prescriptivism and language socialization at Math Corps. *Language in Society, 44*(1), 63-85. https://doi.org/10.1017/S0047404514000748
Ciosek, M., & Samborska, M. (2016). A false belief about fractions – What is its source? *The Journal of Mathematical Behavior, 42*, 20-32. https://doi.org/https://doi.org/10.1016/j.jmathb.2016.02.001
Clawson, C. C. (1994/2003). *The mathematical traveler: Exploring the grand history of numbers*. Perseus.

Cuisenaire, G., & Gattegno, C. (1954). *Numbers in colour: A new method of teaching the process of arithmetic to all level of the Primary School*. Hienemann.

Davydov, V. V. (1962). An experiment in introducing elements of algebra in elementary school. *Soviet Education, 5*(1), 27-37. https://doi.org/10.2753/RES1060-9393050127

Davydov, V. V., & Tsvetkovich, Z. H. (1991). The object sources of the concept of fractions. In V. V. Davydov & L. P. Steffe (Eds.), *Soviet studies in mathematics education. Volume 6. Psychological abilities of primary school children in learning mathematics* (pp. 86-147). National Council of Teachers of Mathematics.

Dougherty, B. J., & Venenciano, L. C. H. (2007). Measure Up for Understanding. *Teaching Children Mathematics, 13*(9), 452-456. http://www.jstor.org.proxy.libraries.rutgers.edu/stable/41198995

Escolano Vizcarra, R. (2007). *Enseñanza del número racional positivo en educación primaria: Un estudio desde modelos de medida y cociente* [Doctoral, Universidad de Zaragoza]. Zaragoza.

Fuchs, L. S., Malone, A. S., Wang, A., Schumacher, R. F., & Namkung, J. (2017, 11 / 01 /). Fraction intervention for students with mathematics difficulties: Lessons learned from five randomized controlled trials [Article]. *Journal of Learning Disabilities, 50*(6), 631-639. https://doi.org/10.1177/0022219416677249

Gattegno, C. (1970a). *Gattegno mathematics text-book 1: Qualitative arithmetic, The study of numbers from 1 to 20*. Educational Solutions.

Gattegno, C. (1970b). *Gattegno mathematics text-book 2: Study of numbers up to 1,000; The four operations*. Educational Solutions.

Gattegno, C. (1970c). *Gattegno mathematics text-book 4: Fractions, decimals, and percentages*. Educational Solutions.

Gattegno, C. (1970d). *What we owe children: The subordination of teaching to learning*. Avon.

Gattegno, C. (1973). *In the beginning there were no words: The universe of babies*. Educational Solutions.

Gattegno, G. C., & Hoffman, M. R. (1976). *Handbook of activities for the teaching of mathematics at the elementary school*. Human Education.

Geary, D. C., Berch, D. B., & Koepke, K. M. (2015). The evolution of number systems. In *Evolutionary origins and early development of number processing.* (pp. 335-353). Elsevier Academic Press. https://doi.org/10.1016/B978-0-12-420133-0.00013-2

Hartwig, M. (Ed.). (2021). *The Tomb Chapel of Menna (TT69): The art, culture, and science of painting in an Egyptian tomb*. The American University in Cairo.

Kieren, T. E. (1976). On the mathematical, cognitive and instructional foundations of rational number. In R. A. Lesh (Ed.), *Number and measurement* (pp. 101-144). ERIC Clearinghouse for Science, Mathematics, and Environmental Education.

Kieren, T. E. (1980). The rational number construct-Its elements and mechanisms. In T. E. Kieren (Ed.), *Recent research on number learning* (pp. 125-150). ERIC Clearinghouse for Science, Mathematics, and Environmental Education.

Lamon, S. J. (1996). The development of unitizing: Its role in children's partitioning strategies. *Journal for Research in Mathematics Education, 27*(2), 170-193.

https://doi.org/10.5951/jresematheduc.27.2.0170
Lortie-Forgues, H., Tian, J., & Siegler, R. S. (2015, 2015/12/01/). Why is learning fraction and decimal arithmetic so difficult? *Developmental Review, 38*, 201-221. https://doi.org/https://doi.org/10.1016/j.dr.2015.07.008
Mack, N. K. (1990). Learning fractions with understanding: Building on informal knowledge. *Journal for Research in Mathematics Education, 21*(1), 16-32. https://doi.org/10.5951/jresematheduc.21.1.0016
Maher, C. A., & Yankelewitz, D. (Eds.). (2017). *Children's reasoning while building fraction ideas*. Sense.
McCrink, K., & Wynn, K. (2007). Ratio abstraction by 6-month-old infants. *Psychological Science, 18*(8), 740-745.
Meert, G., Grégoire, J., Seron, X., & Noël, M.-P. (2013). The Processing of Symbolic and Nonsymbolic Ratios in School-Age Children [Article]. *PLoS ONE, 8*(11), 1-1. https://doi.org/10.1371/journal.pone.0082002
Mendes, J. R., Grando, R. C., Nacarato, A. M., Araujo, J. A. A., Toricelli, L., Penha, P. C., Caporalle, S. M., Soares, I. C., Fonseca, M. A., & Trenthin, P. H. (2004). Números Racionais: aspectos conceituais, o papel da linguagem e dos materiais manipulativos [Rational Numbers: conceptual aspects, the role of language and manipulative materials]. *Horizontes [Horizons], 22*, 53-64.
Morris, A. K. (2000). A teaching experiment: Introducing fourth graders to fractions from the viewpoint of measuring quantities using Davydov's mathematics curriculum. *Focus on Learning Problems in Mathematics, 22*(2), 32-84.
Ni, Y., & Zhou, Y.-D. (2005). Teaching and learning fraction and rational numbers: The origins and implications of whole number bias. *Educational Psychologist, 40*(1), 27-52.
Powell, A. B., & Ali, K. V. (2018). Design research in mathematics education: Investigating a measuring approach to fraction sense. In J. F. Custódio, D. A. da Costa, C. R. Flores, & R. C. Grando (Eds.), *Programa de Pós-Graduação em Educação Científica e Tecnológica (PPGECT): Contribuições para pesquisa e ensino [The Graduate Program in Science and Technology Education (PPGECT): Contributions for research and teaching]* (pp. 221-242). Livraria da Física.
Powell, A. B., Alqahtani, M. M., Tirnovan, D., & Temur, Ö. D. (2022). 'One of three parts, but they are unequal': Elementary school teachers' understanding of unit fractions. *Boletim Grupo de Estudos e Pesquisas em Educação Matemática (GEPEM)*(80).
Roque, T. (2012). *História da matemática: Uma visão crítica, desfazendo mitos e lendas*. Zahar.
Rose, N. J. (1998). *Mathematical maxims and minims*. Rome.
Rosenberg-Lee, M. (2021). Probing the neural basis of rational numbers: The role of inhibitory control and magnitude representations. In W. Fias & A. Henik (Eds.), *Heterogeneous Contributions to Numerical Cognition* (pp. 143-180). Academic Press. https://doi.org/10.1016/B978-0-12-817414-2.00001-4
Schmittau, J. (2003). Cultural-historical theory and mathematics education. In A. Kozulin, B. Gindis, V. S. Ageyev, & S. M. Miller (Eds.), *Vygotsky's educational theory in cultural context* (pp. 225-245). Cambridge.
Schmittau, J., & Morris, A. (2004). The development of algebra in the elementary mathematics curriculum of V.V. Davydov. *The Matheamtics Educator, 8*(1), 60-87.

Siegler, R. S. (2016). Magnitude knowledge: The common core of numerical development [Article]. *Developmental Science, 19*(3), 341-361. https://doi.org/10.1111/desc.12395
Siegler, R. S., Duncan, G. J., Davis-Kean, P. E., Duckworth, K., Claessens, A., Engel, M., Susperreguy, M. I., & Chen, M. (2012). Early Predictors of High School Mathematics Achievement. *Psychological Science, 23*(7), 691-697. https://doi.org/doi:10.1177/0956797612440101
Siegler, R. S., Fazio, L. K., Bailey, D. H., & Zhou, X. (2013, 1//). Fractions: the new frontier for theories of numerical development. *Trends in Cognitive Sciences, 17*(1), 13-19. https://doi.org/https://doi.org/10.1016/j.tics.2012.11.004
Sierpinska, A. (2005). On practical and theoretical thinking and other false dichotomies in mathematics education. In M. G. Hoffmann, J. Lenhard, & F. Seeger (Eds.), *Activity and sign: Grounding mathematics education* (pp. 117-135). Springer.
Snapper, E. (1979, 1979/09/01). The Three Crises in Mathematics: Logicism, Intuitionism and Formalism. *Mathematics Magazine, 52*(4), 207-216. https://doi.org/10.1080/0025570X.1979.11976784
Souza, A. M. V. F. d., & Powell, A. B. (2022). Kyozaikenkyu: Essential lesson planning in Japanese Lesson Study. *Caminhos da Educação Matemática em Revista*.
Steffe, L. P. (2001, 2001/01/01/). A new hypothesis concerning children's fractional knowledge. *The Journal of Mathematical Behavior, 20*(3), 267-307. https://doi.org/https://doi.org/10.1016/S0732-3123(02)00075-5
Struik, D. J. (1948/1967). *A concise history of mathematics* (3rd Revised ed.). Dover.
Thompson, C. A., Taber, J. M., Sidney, P. G., Fitzsimmons, C. J., Mielicki, M. K., Matthews, P. G., Schemmel, E. A., Simonovic, N., Foust, J. L., Aurora, P., Disabato, D. J., Seah, T. H. S., Schiller, L. K., & Coifman, K. G. (2021). Math Matters: A Novel, Brief Educational Intervention Decreases Whole Number Bias When Reasoning About COVID-19. *Journal of Experimental Psychology: Applied, 27*(4), 632–656.
Toledo, R. V. F., Abreu-Mendoza, R., & Rosenberg-Lee, M. (2022). Brazilian math teacher's magnitude representation and strategy use in fraction comparison: A mixed methods study. *Caminhos da Educação Matemática em Revista*.
Torbeyns, J., Schneider, M., Xin, Z., & Siegler, R. S. (2015, 6//). Bridging the gap: Fraction understanding is central to mathematics achievement in students from three different continents. *Learning and Instruction, 37*, 5-13. https://doi.org/http://dx.doi.org/10.1016/j.learninstruc.2014.03.002
Vamvakoussi, X., & Vosniadou, S. (2004). Understanding the structure of the set of rational numbers: a conceptual change approach. *Learning and Instruction, 14*(5), 453-467. https://doi.org/https://doi.org/10.1016/j.learninstruc.2004.06.013
Vamvakoussi, X., & Vosniadou, S. (2010). How many decimals are there between two fractions? Aspects of secondary school students' understanding of rational numbers and their notation. *Cognition and Instruction, 28*(2), 181-209.
Van Dooren, W., Lehtinen, E., & Verschaffel, L. (2015, 2015/06/01/). Unraveling the gap between natural and rational numbers. *Learning and Instruction, 37*, 1-4. https://doi.org/https://doi.org/10.1016/j.learninstruc.2015.01.001
Van Hoof, J., Vandewalle, J., Verschaffel, L., & Van Dooren, W. (2015). In search for

the natural number bias in secondary school students' interpretation of the effect of arithmetical operations. *Learning and Instruction, 37*, 30-38. https://doi.org/https://doi.org/10.1016/j.learninstruc.2014.03.004

Vergnaud, G. (1983). Multiplicative structures. In R. Lesh & M. Landau (Eds.), *Acquisition of mathematics concepts and processes* (pp. 127-174). Academic.

Zhou, Z., Peverly, S. T., & Xin, T. (2006, 10//). Knowing and teaching fractions: A cross-cultural study of American and Chinese mathematics teachers. *Contemporary Educational Psychology, 31*(4), 438-457. https://doi.org/https://doi.org/10.1016/j.cedpsych.2006.02.001