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Epistemological models of mathematical structuralism in the french didactic tradition: networking of theoretical frameworks

ABSTRACT

Abstract: In my research, different theoretical frameworks in the didactics of mathematics are used to study the teaching-learning phenomena related to mathematical structuralism at university level. The aim of this article is to discuss, in the context of this research, the specificity and complementarity of two frameworks from the French didactic tradition - the Theory of Didactic Situations (TSD) and the Anthropological Theory of the Didactic (ATD) - from the point of view of the theorization of knowledge that these frameworks offer and of the epistemological postures that they assume, as well as their consequences for research. It is an opportunity to question the various meanings and uses of epistemology in current didactic research while returning to the sources of experimental epistemology, in the sense of Brousseau. Our corpus of data comes from the work of students engaged in solving tasks on classifying models of an invented axiomatic structure (the theory of banquets), after having learned Group Theory. While the underlying didactic engineering was designed within the framework of TDS, networking with ATD provides complementary insights into the observed phenomena and a space for dialogue between theoretical frameworks.

Keywords: Epistemological models of mathematical structuralism, French tradition of Didactic, Networking of theoretical frameworks

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INTRODUCTION

This article is built around two groups of research questions that are linked but situated on different levels. It is in fact the second group of research questions, related to the epistemological aspects of didactical frameworks and the resulting methodological questions, which is our main focus since it is at the heart of the call for papers in this special issue. The first group of questions will act as a contextualization: as Brousseau (2005, p. 11) points out, “the reciprocal relationship between didactics and epistemology is only really revealed in the intimacy of a research project”.

This first group concerns the teaching-learning of abstract axiomatic structures at the end of the Bachelor at university, in particular Abstract Algebra (Group, Ring and Field Theory), which is a source of persistent difficulties for students (HAUSBERGER, 2020). It is therefore a teaching problem, which I have posed in these terms: how to facilitate the access to structuralist thinking? (HAUSBERGER, 2021) What situations/activities should be proposed to students? Indeed, the didactic issue goes beyond that of a theory applied to objects of different nature (the issue of the so-called FUGS - formalizing, unifying, generalizing and simplifying – concepts; ROGALSKI, 1995), it is polarized on a unified and systematic treatment of axiomatically presented structures: the same type of questions are asked about them and solved with the same type of tools, putting forward the bridges between these structures (HAUSBERGER, 2017). A detailed understanding of the epistemology of mathematical structuralism thus appears to be a necessary prerequisite for didactic action.

Moreover, it is a question of articulating epistemological and didactical analyses within a theoretical framework that is conducive to theorizing and analyzing the didactical phenomena linked to the acquisition of structuralist thinking in learners, hence a second group of questions, situated on a meta-didactical and methodological level: how to model structuralist practices and the processes of acquisition of structuralist thinking within the Theory of Didactic Situations (TDS; BROUSSEAU, 1991) on the one hand and the Anthropological Theory of the Didactic (ATD; CHEVALLARD & BOSCH, 2020) on the other? To what respect do these two theoretical frameworks offer different or complementary perspectives and modeling tools? What are the differences to be noted in the epistemological foundations of these two didactic theoretical frameworks and what are the consequences for the research?

Our meta-didactic reflection around a didactic project is essentially limited in this paper to the French didactic tradition (ARTIGUE & al., 2019) of which TDS and ATD are two pillars. This tradition was born out of a shift from Piaget’s cognitive psychology, where the center of attention is no longer the learner but the situation, conceived in TDS as a system of interactions between three poles: the students, the teacher, and the mathematical knowledge. By theorizing the necessary transposition of mathematical knowledge for it to be taught, ATD highlights the relativization of knowledge and the subjection of persons to the institutions that shape and transmit this knowledge, while developing a general model of human activities that allows for the analytical description of mathematical and didactical practices using the key notion of praxeology (CHEVALLARD & BOSCH, 2020).

The theory of banquets, as a didactic engineering (ARTIGUE, 2014) aimed at providing answers to the teaching issue of mathematical structuralism, has been constructed in support of the TDS. If this conforms to the classical methodology of the French didactic tradition, it was also a question of investigating the problematic question of existence of fundamental situations (in the sense of Brousseau) in the case of FUGS concepts. This constitutes a second justification for the choice of the first theoretical framework, which moreover calls for a discussion of one of its founding hypotheses (a postulate of existence of such situations, or even the very notion of fundamental situation and notably its extension as a concept in the field of didactics). In doing so, the study in historical epistemology (BONTEMS, 2006) aiming at understanding the development of structuralist thinking (around questions, methods and types of problems that characterize it) led to the elaboration of the situations of the theory of banquets and to the experimentation of the didactic engineering, in class with third year Bachelor students of mathematics and in laboratory sessions (with two pairs of Master students). Dialectically, the observed phenomena led to a revisiting of the epistemological analyses in a confrontation between experimental (in the sense of BROUSSEAU, 2005) and historical epistemology (HAUSBERGER, 2022) and to complementary didactic analyses borrowing from other frameworks, e.g., Tall's three worlds (HAUSBERGER, 2023), in the spirit of networking theoretical frameworks (BIKNER-AHSBAHS & PREDIGER, 2014). The possibility of analyzing observed didactic phenomena through the lens of ATD is also raised, which this article aims to explore.

Indeed, a second aspect of my research work concerns the modeling of structuralist practices within the theoretical framework of ATD, in the language of praxeologies. This second approach is motivated by the methodological dimension of mathematical structuralism (an argument of an epistemological nature and linked to the modeling possibilities offered by ATD) as well as an orientation of the questioning towards the study of the conditions and constraints that favor or hinder the development, within educational institutions, of the praxeologies that I have called structuralist (HAUSBERGER, 2016, 2018). Moreover, ATD offers other tools for didactic engineering, around the notion of study and research paths (CHEVALLARD & BOSCH, 2020). As the engineering of the banquets was developed in support of the TDS, these latter aspects will not be addressed and will limit ourselves to discussing, from a reflexive point of view, the specific and complementary aspects that the use of the two frameworks brings to the analysis of the data that was collected.

This program structures the article as follows: I begin by discussing the modeling of structuralist thinking in TDS, around the didactic engineering of banquets and its a priori and a posteriori analyses, then present the modeling from the perspective of structuralist praxeologies in ATD, with the aim of shedding light on the same data, resulting from the experimentation of a situation from the banquets. The paper ends with meta-didactic and methodological conclusions, regarding the epistemological dimensions of the two frameworks mobilized and its consequences for research.

DIDACTICAL SITUATIONS FOR MATHEMATICAL STRUCTURALISM: FROM HISTORICAL TO EXPERIMENTAL EPISTEMOLOGY

Our starting point is an epistemological analysis of mathematical structuralism as reported by historians and philosophers (CORRY, 1996; BENIS-SINACEUR, 1987; PATRAS, 2001), for example: “The essence of the method is to abstract from the objects studied their formal substance, in the same way as the process of transcendental abstraction extracts concepts from their empirical roots [...]: from diverse and sometimes unrelated situations, to derive concepts, universal structures, which can simultaneously deal with questions in a priori distinct domains. […] This generality is not in vain, as long as the mathematician gains in lucidity and understanding” (Patras, 2001, p. 57, our translation).

Such accounts place at the heart of mathematical structuralism the movements of abstraction which found it and which Cavaillès has called idealization and thematization (BENIS-SINACEUR, 1987). Moreover, Corry (1996) emphasizes the architectural dimension of the notion of structure, which belongs to the image of knowledge rather than to the body of knowledge. In Bourbaki's manifesto (1998), the idea of structure appears first of all as a concept regulating mathematical thinking, in metaphorical or programmatic form, to designate an architecture hidden behind objects or mathematical theories. Then, it is formalized in a precise and rigorous way in terms of particular structures like groups, vector spaces or topological spaces. Bourbaki insists on the methodological dimension of structuralist thinking and speaks of structures as the “Taylor system” of mathematics. These elements allow us to identify structuralism as a specific form of advanced mathematical thinking, a particular epistemology of mathematics, and a methodology.

This explains why our teaching-learning problem is posed from the point of view of the acquisition of structuralist thinking. In terms of situations, it is a question of identifying which classes of problems make it possible to restore the meaning of structuralist approaches and to allow the deployment of structuralist thinking, in its two movements of abstraction. In methodological terms, if the theory of didactic situations allows to pose the engineering work under the angle of the construction of idouane situations (fundamental, as Brousseau would say, as we will come back to it) and offers the tools to think and manage the didactic situation as a functioning of a system (the didactic triangle), the conceptualization issues linked to structuralist thinking leads us to articulate TDS with an epistemological and semio-cognitive framework, which I have developed specifically for mathematical structuralism and called the “dialectic of objects and structures” (HAUSBERGER, 2017). In other words, I felt it was essential to address this cognitive and conceptualizing dimension, which isn’t covered by the systemic approach of TDS. For the same reasons, it is common in the French didactic tradition to articulate TDS and the Theory of Conceptual Fields (TCF; VERGNAUD, 1990), the third pillar of this tradition. Although concepts-in-action and theorems-in-action (in the sense of Vergnaud) have been identified in analyzing learners’ work on a mathematical forum online (HAUSBERGER, 2016), I have not to date developed a notion of structuralist scheme in Vergnaud’s sense. The didactic engineering needs and the development of a priori analysis tools has led us more towards elements of model theory in order to think from an epistemological point of view about the articulation between objects and structures, which is thus similar to a dialectic between syntax (the structures) and semantics (the objects), and towards Duval’s semio-cognitive framework (DUVAL, 1995). The latter is used in order to take care of the semiotic aspects of the conceptualization processes, the mathematical objects being assimilated to classes of semiotic representations modulo the operations (conversions between semiotic registers and treatments in the sense of Duval) which preserve the objects. These notions are thus articulated with the mathematical notion of isomorphism, in order to think epistemologically and didactically about the conceptualization of an abstract mathematical structure in the form of its different isomorphism classes of models. This is the whole philosophy of the dialectic of objects and structures (HAUSBERGER, 2017), which we now need to implement concretely around the notion of banquet.

A banquet is a set E endowed with a binary relation R which satisfies the following axioms: (i) No element of E satisfies xRx; (ii) If xRy and xRz then $y = z$; (iii) If yRx and zRx then $y = z$; (iv) For all x, there exists at least one y such that xRy. The banquet structure possesses a large variety of models since the system of axioms may be interpreted in quite different settings, beginning with the empirical interpretation of guests sitting around tables (whence its name): xRy if x is sitting on the left (or right) of y. Other domains of interpretation include Set Theory (the binary relation is represented by its graph), Functions ($xRy \Leftrightarrow y = f(x)$ defines a function f according to axioms (ii) and (iv); the other two axioms mean that it is injective without fixed points), Permutation Groups (f is a bijection when E is finite, in other words a permutation without fixed points) or even Matrix Theory (the relation is seen as a function $E^2 \rightarrow \{0,1\}$ and represented by the corresponding matrix; the axioms express rules on the number of 1 in each row and column) and Graph Theory (xRy if and only if the vertices x and y are connected by an edge oriented from x to y). The structure theorem of banquets (any banquet is the disjoint union of “tables”, the cyclic banquets) thus corresponds to the well-known theorem of canonical cycle-decomposition of a permutation, but the analogy remains hidden since the binary relation of banquets is different from binary operations that define groups. The learning goals are made explicit to the students in the beginning of the worksheet in the form of a meta-discourse: “The theory of banquets won’t be found in Algebra textbooks: it is a didactical invention. Its aim is to provide an adequate context to discuss, on a simple example, how a mathematical structuralist theory works...”. The main prerequisite is a course in Group Theory (GT), so that students have already encountered similar structuralist questions and results that will be thematized in the context of banquets.

In part I of the worksheet, students are asked the following questions:

1 a. Coherence: is it a valid (non-contradictory) mathematical theory? In other words, does there exist a model?

1 b. Independence: is any axiom a logical consequence of others or are all axioms mutually independent?

2 a. Classify all banquets of order $n \leq 3$

2 b. Classify banquets of order 4

2 c. What can you say about $\mathbf{Z}/4\mathbf{Z}$ endowed with $xRy \Leftrightarrow y = x+1$?

2 d. How to characterize abstractly the preceding banquet (that is, how to characterize its abstract banquet structure among all classes of banquets, in fact how to characterize its class)?

The abstract/concrete relationship is reversed in part II of the worksheet, which begins with the empirical definition of a table of cardinal number *n* as a configuration of *n* people sitting around a round table. Its aim is to prove that any banquet decomposes as a disjoint union of tables (the “structure theorem”). Altogether in the language of TDS, the theory of banquets decomposes into 4 main (sub-)situations (see HAUSBERGER, 2021 or 2023 for complete statements of tasks): the logical analysis of the system of axioms (I 1); the classification of banquets of small cardinal numbers (I 2); the axiomatic definition of tables (II 1); the structure theorem (II 2). In the sequel, we will mainly focus on I 2 (the classification situation).

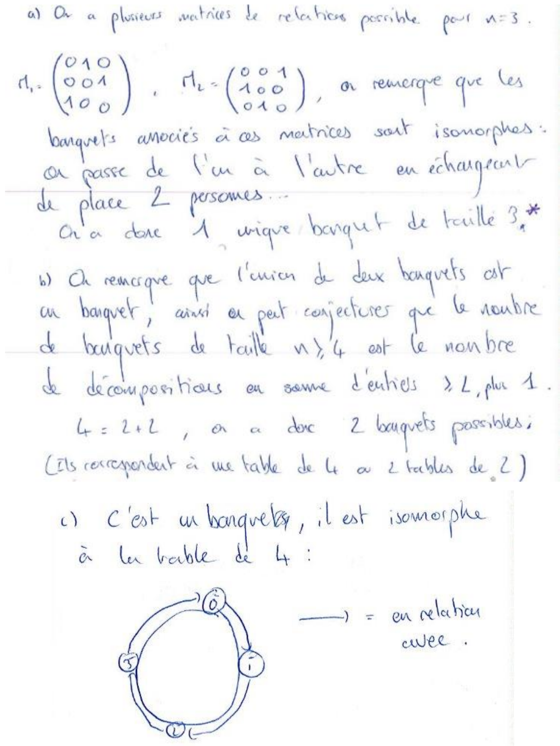
In his glossary, Brousseau (2010) defines a *fundamental situation* as a “schema situation capable of generating, by variation of the didactical variables which determine it, the set of situations corresponding to a determined piece of knowledge (savoir)”. While TDS implicitly posits the existence of a fundamental situation for any piece of knowledge, the construction of fundamental situations in the case of FUGS concepts (ROGALSKI, 1995) is debated in the mathematics education research community (BOSCH & al., 2018). In (HAUSBERGER, 2021), I argued that banquet tasks are situations more characteristic of mathematical structuralism than fundamental in the sense that the choice of the different didactic variables would allow the learner to associate the knowledge aimed at with the conditions that could justify it and make it necessary. From the point of view of epistemological argumentation, it can be refered to the work of Marquis (2014, 2016) who highlights four key moments of the so-called abstract method: a first moment of constitution of a “domain of meaningful variation” (at least three distinct types of objects share invariant characteristics); a second moment of formalization of the particular objects where their semantics are abstracted to deal with them formally; a third moment where the invariant properties are abstracted and presented in an autonomous way (the axiomatic method is a good tool for this); finally, a last moment where an identity criterion is fixed (what is the natural notion of isomorphism, in the given category of objects). The domain of variation can then continue to expand but the attention is quickly directed towards internal problems such as classification or decomposition into elementary substructures (the "structure theorems"). With reference to Marquis' four moments, part I of the banquets leads to the creation of the "domain of meaningful variation" by contrasting empirical models, the example (**Z**/4**Z**, *R*) from the worksheet and other possible mathematical models. This is the inverse direction of the historical process where abstract constructs and axiom systems are obtained out of various examples that precede them. The formalist moment and the moment of presentation are partially implemented during the task of axiomatic definition of tables (II.1). The fourth moment is divided into the classification task (I.2) and that of theoretical elaboration (II.2). Reasons to organize in this order the sequence of situations (e.g. the economy of didactic time) and the values of other didactic variables (type of structure, semiotic registers introduced by the teacher to enrich the milieu, etc.) are discussed in (HAUSBERGER, 2021).

To further discuss potential didactic issues related to the under-determination of the target knowledge by the situation, let us give a short a priori analysis of the classification situation of the banquets. In the case of three elements *x*, *y*, *z*, we can assume up to permutation of elements that *xRy* (under i) and iv)); necessarily, *(yRx* or *yRz*) and (*zRx* or *zRy*), again under i) and iv). Of the four cases, only *yRz* and *zRx* is possible, by virtue of axioms ii) and iii). The reasoning is similar with four elements, but it requires repeating several times the "up to permutation" argument. This leads to two classes: *xRy*, *yRx*, *zRt*, *tRz* and *xRy*, *yRz*, *zRt*, *tRx*. One may expect students to stop at this stage, while it remains to justify that these two classes are distinct (and nonempty, by providing a model). This requires the notion of isomorphism, in fact the knowledge of properties invariant under isomorphism. In the case of groups of order 4, well-known to students, the presence or absence of an element of order 4 is usually invoked. Working out the analogy between GT and banquets consists in identifying a pattern of cyclicity: the notion of order in GT corresponds to the cardinal of the "chain" generated by an element (by iteration of the relation), which is a closed loop in the case of finite cardinality. One expects the cyclic pattern to be recognized, the more so as it is suggested by the mental image behind banquets. The aim of questions c) and d) is to lead students to make this mental image explicit and formalize a notion of cyclic banquet. In another more semantic approach, matrix or graph theory may be used to produce generic models that represent all possible cases. Reasoning on axioms may thus be replaced by filling tables with 0 and 1 or drawing arrows between labeled vertices. The graph register is the most powerful to differentiate classes: treatments within this semiotic register to change directions of arrows and then abstracting labels may suggest isomorphism classes, in the etymology of isomorphism (having the same shape). In the case of cardinal 4, this should be helpful to identify either the cyclic graph or the graph composed of two components of two elements connected by a double arrow each.

The milieu (in Brousseau's sense) has antagonistic aspects: for example, a metadiscourse on abstraction included in the worksheet is antagonistic to reducing the classification to the case of empirical banquets. Students are expected to define the notion of banquet isomorphism $\varphi:(E,R)\to(E',R')$ as a bijection $E\to E'$ that preserves relations: $xRy \Leftrightarrow \varphi(x)R'\varphi(y)$. But the necessity, due to the characteristics of the situation, of such a formalization cannot be asserted. The potential of the situation to lead students to thematize GT concepts according to the structuralist methodology requires being subjected to contingency via experimentation.

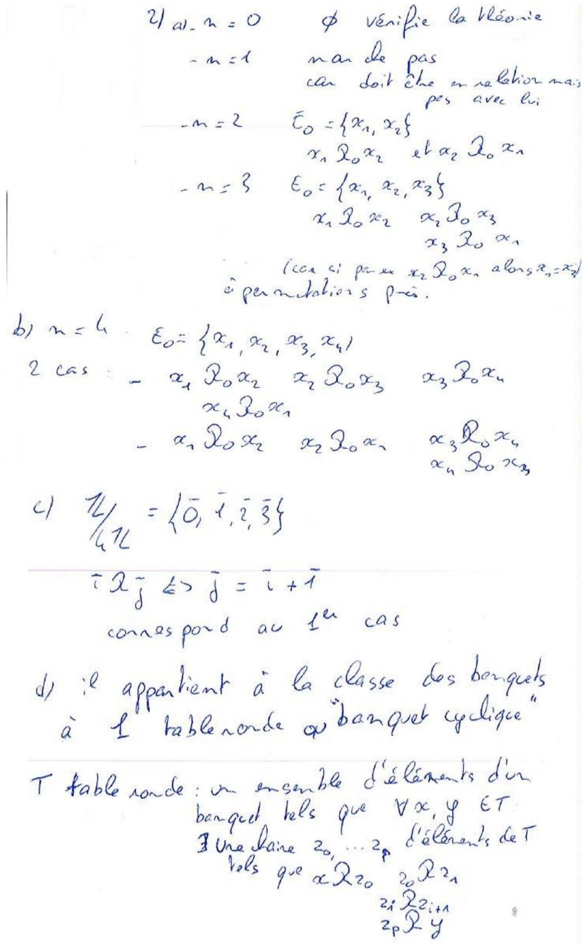
Let us now present some elements of a posteriori analysis. We will first discuss the work of two groups of four third year university students that took place during a classroom experimentation of the banquets. The numbering of the groups is consistent with (HAUSBERGER, 2021) where the reader may find a more detailed discussion of the data.

Figure 1 - Written work of group 3



Group 3 students notice that the banquets associated to the matrices M_1 and M_2 are isomorphic by referring to “exchanging places between 2 persons”. The notion of isomorphism, not formalized, is thus interpreted semantically. The (not formalized) identification of the operation of union of banquets (“the union of two banquets is a banquet”) allows them to conjecture the correct number of isomorphism classes in the general case of n elements: a class is characterized by the number of persons per table (“The number of decompositions into sums of integers greater than or equal to 2 plus 1”). The example $(\mathbf{Z}/4\mathbf{Z},R)$ is converted (in the sense of Duval) from the register of functions to that of empirical banquets, passing through the register of graphs so that the circular banquet of cardinal 4 is recognized by a visual process of pattern recognition.

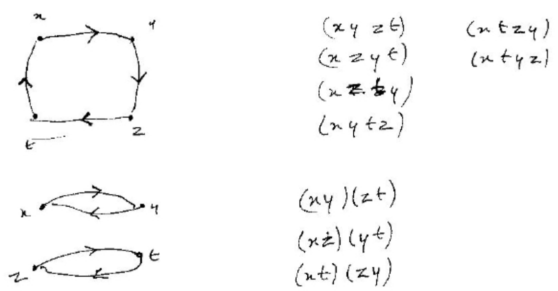
Figure 2 - Written work of group 5



By contrast, the students in group 5 reason directly from the axioms. The word class only appears in their answer to question d), when it is used explicitly in the worksheet. The cyclic banquet is identified and characterized abstractly by the possibility of connecting any two elements by a chain (iteration of the relation). The students were probably not aware before that they were performing a classification up to isomorphism, this fact being hidden behind the algebraic symbolism and reasoning by permutation of letters.

In order to deepen our understanding of how the situation works with respect to the abstraction processes at play and the obstacles to formalizing a notion of banquet isomorphism, I conducted laboratory sessions with Master students. We take up the names used in (HAUSBERGER, 2023) and start with a first pair (Chris/Debby), which produced a classification based on the graph register (Figure 3).

Figure 3 - Chris/Debby's classification of banquets of cardinal 4



This leads them to count 9 banquets, on which the group exercises a critical eye:

Chris: There would be 9 of them.

Debby: Nevertheless, we only considered objects that we know. But since the beginning, we have been talking about a structure.

Chris: But wait, the elements can always be numbered. What could go wrong?

Debby: Our own consistency.

Chris: But here, we thought about relationships, we didn't think about the objects themselves, we didn't take a particular relation.

Debby: Never mind.

The students show a good mastery of the formal method, based on symbolism: they have performed an abstraction of the nature of the objects and the semantics of the relation. However, they do not reach the stage of the expected new identity principle, which is crucial in the abstract method according to Marquis. Another retroaction of the milieu thus proves to be necessary, and takes the form of an intervention of the instructor I:

Debby: So there would be 2 classes up to isomorphism, this kind of object and this kind.

Chris: There, **Z**/4**Z** and there **Z**/2**Z** x **Z**/2**Z**, in fact.

I: Are you thinking about the classification of groups?

Debby: Necessarily, we think about the classifications we know.

I: So there are 2 types of objects and here you have listed them all on *x*, *y*, *z*, *t* [...] You have listed all the possible oriented graphs on *x*, *y*, *z*, *t* that fulfills the axioms. [...] And why do you say there are two classes?

Chris: Two classes? We have put all the permutations behind, anyway.

I: And why would (*x y z t*) and (*x y t z*) be the same?

Chris: No, not the same, of the same type.

I: What does it mean to be of the same type?

Chris: I am thinking of permutations. One will loop faster than the other. I am clearly thinking about the order behind it.

Debby: A bijection. One can pass from one element of this class to another by a bijection, but not between the 2 classes.

I: Isn't it always possible to find a bijection between two sets of same cardinal number?

Chris: Yes it is!

Debby: Ah yes, but will it respect the structure? […]

In (HAUSBERGER, 2022), I draw a parallel between the historical difficulties to the emergence of the abstract method in mathematics and what the experimental genesis has produced in a teaching-learning environment with banquets. Such rapprochements (while taking into account the differences) are at the heart of the emergence of the field of didactics (in the French tradition), whose first name given by Brousseau was "experimental epistemology". Nevertheless, the role played by social interaction with the teacher, bearer of the norm of knowledge, in making the notion of banquet isomorphism emerge, calls for a greater consideration of the sociocultural dimension of learning, which also proceeds by acculturation. The notion of fundamental situation, taken in its idea of epistemological necessity, tries to evacuate in a certain way this facet of learning (despite the notion of institutionalization). Let us now see what the institutional approach of ATD may bring to our understanding of the didactic phenomena that have been unveiled.

STRUCTURALIST PRAXEOLOGY: COMBINING EPISTEMOLOGICAL AND INSTITUTIONAL APPROACHES

The structuralist methodology, described by Bourbaki in its Manifesto (*The Architecture of Mathematics*; BOURBAKI, 1998), is the epistemological anchor-point of the notion of *structuralist praxeology* as an epistemological model of mathematical structuralist practices. The ATD framework is used, which "posits that any human activity can be decomposed into a succession of tasks of various types" (CHEVALLARD & BOSCH, 2020), hence the theory of praxeologies. In praxeological terms, the structuralist method consists in the passage from a praxeology **P** = $[T/?/?/\Theta_{particular}]$ where it is unclear which technique τ to apply, to a structuralist praxeology $\mathbf{P_s} = [T^g/\tau/\theta/\Theta_{structure}]$ where, modulo generalization of the type of tasks (T^g), the theory of a given type of structure guides the mathematician in solving the problem. As a paradigmatic illustration in the context of Abstract Algebra, a thread from an online forum has been analyzed (HAUSBERGER, 2018). In order to prove that the ring of decimal numbers is a principal ideal domain - PID (T: show that a given ring is a PID), a forum participant proposes to prove that any subring of **Q** is a PID (T^g: show that any subring of a given PID is a PID), with the (erroneous) structuralist technique τ: show that any subring of a PID is again a PID. The underlying structuralist technology is the idea that striking features of rings are preserved by taking subrings, which unfortunately isn't true for being a PID (**Z**[X]⊂**Q**[X] is a counter-example). The objects-structures dialectic thus gives rise to abstract tasks (e.g. t=τ) that are meant to provide new powerful technologies to solve tasks of type T (here, showing that the given ring is included in a PID). In other words, the objects-structures dialectic subsumes both a dialectic of the particular and the general, and of the concrete and the abstract, which are interrelated.

Before applying it to Group Theory and banquets, let us discuss the status of the model of structuralist praxeologies with respect to didactic research. It is an epistemological model in the sense of Florensa et al. (2016) who also envisage relationships between epistemology and didactics in terms of mutual enrichment of both fields: "On the one hand, considering teaching and learning phenomena as part of the empirical basis of epistemology enables proposing new epistemological models of mathematical bodies of knowledge. On the other hand, these epistemological models provide guidelines for the design and analysis of new teaching proposals, which, in turn, show the constraints coming from the spontaneous epistemologies in school institutions." Faithful to the French tradition, ATD thus takes up Brousseau's idea of experimental epistemology while underlining the relativity of knowledge to the institutions that develop and teach it, from which result a set of conditions and constraints to be analyzed in order to understand the didactic phenomena of diffusion of praxeologies. Precisely, Florensa et al. define a Reference Epistemological Model as an "alternative description of a body of knowledge elaborated by researchers in order to question and provide answers to didactic facts and problematic aspects taking place in a given institution". A structuralist praxeology is a model, an ideal type, elaborated by considering the scholarly knowledge and its epistemology, which raises the issue of its pertinence to describe didactic phenomena related to university as a teaching institution and the type of phenomena that it may reveal.

Our goal is to use this model in order to understand what it means for students to thematize their GT knowledge in the tasks of classifying banquets of small cardinality. Praxeologies are a good tool since, in ATD's cognitive paradigm, relations students have to mathematical objects, in their institutional positions as students, emerge from the relations they have to the praxeologies that put the object in use in some of the components (technique, technology, or theory). The ATD provides a finer grain of analysis than what we described in the a priori analysis of banquets outlined above. We therefore need to investigate students' praxeological equipment in GT, in particular regarding the type T of tasks t_n:

classify groups of order n (for small values of n). Unfortunately, we do not have at hand the GT material that the students who classified banquets worked with in previous abstract algebra courses, and these students also come from various horizons (different post-secondary institutions). But recent research (BOSCH & al., 2021) emphasized that university curricular were more or less stable due to the diffusion of a common culture worldwide, transmitted through standard textbooks. This culture may also be investigated through mathematical forums online in a similar way to what I have done for some notions in Ring Theory (HAUSBERGER, 2016, 2018).

Thus, different documents can be found on the internet, which can be related to different stages of GT learning. The Master's level handout https://agreg-maths.univ-rennes1.fr/journal/2008/Petitsgroupes.pdf, which outlines systematic methods for solving t_n for n<16, exemplifies a culmination of praxeological development, while the forum https://les-mathematiques.net/vanilla/index.php?p=discussion/2320758#Comment_2320758 questions ways of solving the task t_4 at a more elementary level, without using the notion of order of an element. The context of the question is not specified, but the participants of this forum obviously know more advanced techniques that they do not wish to use in order to return to elementary techniques that have probably been forgotten (phenomenon of obsolescence of praxeologies in favour of more efficient techniques). Conversely, the forum https://www.ilemaths.net/sujet-groupes-d-ordre-4-484690.html shows a beginner learner attempting to solve t_4.

The analysis of these various sources allows us to conjecture that the institutional relationship to the type of tasks T of a student who has assimilated GT is generated by a regional praxeology that unifies different types of tasks, namely T_1: prove that $(G,*)$ is a group, T_2: compute the order of an element of the group, T_3 : show that 2 groups are isomorphic, T_4: show that 2 groups are not isomorphic, T_5: show that a group is cyclic, T_6: show that a group is isomorphic to the direct product of two cyclic groups, T_7: show that a group is Abelian, T_{ab}: classify finite abelian groups (tasks indexed by the cardinal n). The notion of order is an essential element of praxeologies and the classification theorem of finite Abelian groups is one of the most salient structuralist elements of the logos. Moreover, the cardinal of the direct product of two groups is the product of the cardinals, which invites to relate the prime factors decomposition of the cardinal number n to the form of the group decomposition, which mathematicians wish to make "canonical". The structuralist technique τ for solving T in the non-Abelian case thus involves the examination of the local generalizations $n=p$ (prime number), $n=p^2$, $n=pq$ (product of two primes), $n=p^2q$, which the available conceptual tools allow to solve. This explains why the university teaching institution limits the study of T to n<16, because the classification of finite (simple) groups, in its generality, is out of reach at Master level. On the side of elementary methods, we note τ_{elem}: determine the Latin square (Sudoku) tables of size n on n letters e (the neutral element), $x_1,...,x_{n-1}$ since the table of a group is a Latin square. As for matrices in the case of banquets, using the properties of Latin squares is an alternative to direct reasoning on axioms, and then the notion of order is the key notion to differentiate the isomorphism classes of corresponding groups.

Let us now return to the theory of banquets in order to show how these new contributions of ATD allow us to shed light on the didactic phenomena encountered. The banquets have also been submitted to a pair of advanced Master students. Alice has a PhD in theoretical physics and is preparing for the agrégation of mathematics (competing examination to become a upper or post-secondary teacher). Let us examine, under the light of our praxeological analyses, how the pair performs the classification task, without any help:

Bob: Cardinal number 3…

Alice: The circular thing, people a,b,c around the table. (a,b),(b,c),(c,a). It remains to be seen that this is the only one. (a,b) by numbering, it is still valid.

Bob: (a,c),(c,b),(b,a)?

Alice: It’s the same model, up to isomorphism.

Bob: That’s true.

Alice: (b,a)... there's going to be a problem, because c is going to be sent on what? If c is sent on a or b, as a and b are already reached, we will deny (ii).

Bob: If we had (a,b) and (b,a) we wouldn't know what to do with c…

Alice: Yes, that’s it. Because his two potential right-wing neighbors already have one neighbor

Bob: So it’s necessarily (b,c) and we complete.

Alice: Perhaps cardinal number 4 will be more interesting. Shall we say {a,b,c,d} ?

Bob: Yes.

Alice: So there is the circular model... are you following me?

Bob: Always... but, in this case, there can be several if you put them a,b,c,d around a table…

Alice: Yes, but you’ll be able to find a bijection, which amounts to a renumbering. If you want, the natural morphisms in there will be... is there a way to send E on E' by a bijection that sends R on R'? So if you have a circular model, you’re going to be able to send it on a circular model by a permutation.

Bob: Uh, yes…

Alice: So we always have (a,b); we always have (b,c)... ah, can b send itself to a? That would make a first case separation.

Bob: It would make a two-table banquet, so to speak.

Alice: Yes, this is a possibility. You can have (a,b),(b,a),(c,d),(d,c). In fact, we’re going back to the previous banquets. We have the circular banquet $R_{C,4}$, and we have, one could say, finally a direct sum in fact. It is a direct sum of banquets: $R_4=R_2\oplus R_2$. Are there others? I don’t think so.

Bob: Are there other possible direct sums? No, because there is no one-person banquet.

Alice: In theory, you can have irreducible models, which do not break down into direct sums, and which are a priori different from the circular model. But here, if we have (a,b) and if we put (b,a), then the rest is specified; so we will try to put (b,c). If we put (c,d) we fall back to the circular banquet; (c,a) we're screwed. So this is the only possibility, I don't know if you follow me…

Bob: OK, so we have our two models.

This dialogue is striking on several aspects. The students start with an elementary technique similar to the classification of groups (direct reasoning on axioms), then quickly assimilate the mental image of banquets, which directs them to the notion of cyclicity. Alice then poses a notion of isomorphism as a principle of identity, at the request of Bob who acts as an antagonistic element in the milieu and is somehow stuck at the same stage as Chris and Debby (formalist phase of the abstract method). Moreover, Alice poses a notion of direct sum of banquets, undoubtedly by analogy with the classification of Abelian groups, and thus anticipates the following situations of banquets. Now our regional praxeological model of GT allows us to explain Alice's rationality: one finds in Alice's discourse the most salient elements of the structuralist logos, transposed to the case of banquets with great dexterity, while Bob manifests on his side the necessity of progressively reconstructing the punctual praxeologies in the case of banquets before unifying the whole in a regional praxeology analogous to that of GT, as well as – in fine - the two regional praxeologies (from GT and banquets) in a global praxeology whose logos is the structuralist methodology in its generality.

The praxeological model also sheds light on the classroom experimentation: elementary Sudoku-based techniques or reasoning on axioms can be found in the student groups' productions and are well illustrated by groups 3 and 5. Group 3 also assimilated and transposed some of the more advanced structuralist logos centered on structure theorems, even recognizing the additive nature of the banquet problem (compared to groups where the cardinals are multiplicative with respect to the product). The mental image seems to have played an important role for the transposition of these praxelogical elements from GT to the banquets, in view of the students' discourse (changing the place of the persons, tables, etc.). However, these elements of logos play more the role of a heuristic: the students affirm the results without elaborating the proofs, thus without transposing the praxis of GT beyond the elementary praxeologies. The praxeological development is under construction and the notion of isomorphism of banquets remains the cornerstone, which does not manage to detach itself from the empirical substrate of banquets. We do not have enough elements to elucidate this resistance. One hypothesis to be explored is that of the didactic contract in relation to the topos (in the sense of the ATD) of the students in GT, and in the learning of abstract structures at university in general. The appropriate notion of isomorphism is most of the time given in conjunction with the axiomatic definition of the structure (e.g. GUIN, 1997, Chap. I, p. 16). It is seldom questioned and problematized; its nature as an identity principle is often lost sight of in the complexities of the more general notion of homomorphism (HAUSBERGER, 2017). This necessarily influences the students' relation to the notion of isomorphism and hinders the possibilities of unification and transfer from one structure to another of the reasoning principles that make use of this notion.

CONCLUSION

The research on the teaching-learning of mathematical structuralism at university level that is presented in this paper is fully situated in the French didactic tradition by the role assigned to epistemological analyses, in the analysis of didactic phenomena and in the work of didactic engineering. It is the occasion to distinguish the different uses of the term epistemology which, starting from a field of research at the crossroads between history and philosophy of sciences, is adjectivized in the didactic literature to designate different types of models (a fundamental situation in the sense of TDS, an epistemological/praxeological model of reference in the sense of ATD) elaborated by didacticians to account for the functioning of knowledge in didactic contexts, or even the representations of this knowledge that the different actors of the didactic institutions have (the epistemology of the student, of the teacher, or of an institution itself). The researcher in didactics is then an "anthropologist of knowledge", who deciphers and models practices and their discourses in scholarly contexts as much as in online forums or in classrooms, and then organizes encounters with this knowledge in contexts controlled by theoretical constructs, in order to test hypotheses on the experimental genesis, i.e. in the context of such experimentation, of this knowledge (versus its historical genesis). It is therefore understandable that the development of this research methodology suggested to Brousseau the name of "experimental epistemology" to designate the field of didactics, at the time of its foundation in rupture with the genetic epistemology of Piaget.

Both the TDS and the ATD evacuate, in a certain way, the strictly cognitive aspects, which led us to complete TDS with an epistemological and semio-cognitive framework (the object-structure dialectic) to elaborate the didactic engineering of banquets. Some elements of this framework have been sketched and the reader can refer to the corresponding articles for more details. In ATD, the new dialectic that have been introduced, that of objects and structures, finds a praxeological expression that allows for the inclusion of its main crucial aspects in ATD. I give it the status of a dialectic of study and research processes, in the same way as those put forward by Chevallard (systems and models, as far as mathematization is concerned, or media and milieus and other dialectics that organize study and research), when the study of a mathematical problem is conducted in a structuralist perspective. This formalization also allows further development of the methodological dimensions of the objets-structures dialectic (as a set of techniques and discourse on these techniques), in comparison with the initial semio-cognitive framework which dually places more emphasis on the processes of abstraction and conceptualization. TDS also has its own contributions, through the notion of didactic contract, for example, and the phenomena of the constitution of meaning for a learner, which is played out and read through the complex game governed by all the didactic variables of the situation.

The readings that TDS and ATD allow of our data are therefore not the same. The networking of theoretical frameworks is not limited to a crossing of different perspectives on the same data. The data are intimately linked to the theoretical frameworks within which they were collected. For example, the ATD analysis of the data collected on banquets required additional data collection in the form of a survey online, that is the collection of various materials that could be used to draw up a praxeological model of the institutional relationship to the task of classifying groups of small cardinal. It was therefore necessary to enrich and reshape the data from the banquets before an analysis in terms of structuralist praxeologies could be carried out.

The benefits of the networking were not limited to increasing the understanding of the didactic phenomena linked to the thematization of structuralist notions of group theory in the case of banquets. We can also note a benefit at the theoretical level: on the one hand, the notion of structuralist praxeology comes from the works on the dialectic of objects and structures, on the other hand the epistemological resistance to the emergence of the concept of isomorphism which appeared in the situation of banquets led us to study the problem under the institutional angle of ATD. This direction led us to pose the thematization problem in terms of the transfer of a regional group-theoretic praxeology to the banquet setting, supported by a structuralist logos under construction, and to interpret the success of the advanced students as a unification of regional structuralist praxeologies related to two different structures into a global structuralist praxeology. This rewriting of both phenomena and theories emerged from the networking practices.

Finally, we can also note the methodological benefits of the dialogue between the two theories. The plurality of possible models of knowledge, in terms of situations or praxeologies, leads us to question both the spectrum of empirical data that these models can cover (which led us to extend or supplement both TDS and ATD theoretical constructs), the methodologies of construction of these models (based on historical epistemology), and the foundations that allow their articulation both with epistemology as a discipline and with the theoretical constructs of didactics. This questions in particular the notion of fundamental situation (FS) in TDS and that of epistemological/ praxeological reference model (E/PRM) in ATD. If the E/PRM relativizes the notion of knowledge model to a researcher's position and to the institutions that are involved in relation to the FS, the validity of the model is nonetheless subject to contingency, i.e. to its confrontation with experimental epistemology in Brousseau's sense. In doing so, the notion of a-didactic situation in the sense of Brousseau naturally comes to the forefront, which leads the ATD to reflect on the theoretical forms of its integration within the ATD framework and on its consequences with respect to the methodology of elaboration and experimentation of the study and research paths. These are some of the avenues that make sense within the framework of our research on structuralism, but which also find an echo more generally in the works carried out in ATD and TDS, as well as through other theoretical frameworks in dialogue with these frameworks, thus continuing to vivify the French didactic tradition while contributing to its diffusion on the international scene.

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