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ARTIGO ORIGINAL/ ORIGINAL ARTICLE

Analysis of a University Student's Construction of a Praxeology for Integration Tasks

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ABSTRACT

This study utilizes the model of didactic moments from the Anthropological Theory of the Didactic to examine a university student's praxeological work pertaining to a set of four integration tasks. The research focuses on themes related to the Fundamental Theorem of Calculus (FTC), which elucidate the technological aspects of the employed techniques. By analyzing the integral calculus content in the textbook used by the student at upper secondary level, the study identifies a potential causal relationship between the textbook's treatment of integration and shortcomings in the student's praxeological equipment. Specifically, it is found that an essential element of the FTC—interpreting area in terms of a function—was missing in the textbook's logos block on integration. Furthermore, the analysis reveals a predominance of algebraic techniques over graphical ones in the student's performance of the set of tasks. The findings underscore the impact of educational resources on students' praxeological equipment and highlight the need for a critical evaluation of these resources using didactic transposition analysis. The study advocates developing a praxeological organization for integral calculus in order to guide curriculum designers and textbook authors, thereby bridging the educational gap between secondary and tertiary education and enhancing students' comprehension of integral calculus.

Keywords: Area function, Logos block, The fundamental theorem of calculus.

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INTRODUCTION

Infinitesimal calculus, developed independently in the late 17th century by Newton and Leibniz, was created to solve four major problems (KLINE, 1972): (1) given the formula for the displacement of an object as a function of time, to find its velocity and acceleration at any instant (i.e., the study of motion); (2) to find the tangent to a curve (e.g., used in the study of optics); (3) to find the maximum and minimum of a function (e.g., max/min distance of a planet from the sun); and (4) to find the length of a curve (e.g., the distance covered by a planet in a given period of time). The first two problems are solved by differential calculus and the last two by integral calculus, two branches that are connected by the Fundamental Theorem of Calculus (FTC). Here is a formulation of the FTC, paraphrased from Neuhauser (2011, p. 295, 302):

> Part 1: If f is continuous on $[a, b]$, then the function F defined by
>
> $$F(x) = \int_a^x f(t)dt\,,\ x \in [a, b]$$
>
> is continuous on $[a, b]$ and differentiable on (a, b), with $F'(x) = f(x)$; that is, F is an antiderivative of f.
>
> Part 2: If G is any antiderivative of f on $[a, b]$, then
>
> $$\int_a^b f(x)dx = [G(x)]_a^b = G(b) - G(a).$$

The first part of this theorem links antiderivatives and integrals, and the second part provides a method for computing definite integrals.

In the early 19th century, certain properties were known about the basic concepts of analysis: limits, convergence, continuity, derivatives, and integrals. It was Cauchy, followed by Riemann and Weierstrass, who provided a rigorous foundation for the calculus, using the already-existing algebra of inequalities (GRABINER, 1983). The construction of the real number system was, according to Edwards (1979), the most important step in the arithmetization of analysis during the late nineteenth century. The final loose end was tied up by Weierstrass in his purely arithmetic formulation of the limit concept involving only real numbers, without references to motion or geometry: $\lim_{x \to a} f(x) = L$ provided that, given $\varepsilon > 0$, there exists a number $\delta > 0$ such that $|f(x) - L| < \varepsilon$ if $0 < |x - a| < \delta$ (EDWARDS, 1979, p. 333).

The studies done by Radmehr (2016) and Thompson and Silverman (2008) show that students could apply the FTC for finding the definite integral, but they did not have any conceptual knowledge about it. From his historical reflections on teaching of the FTC, Bressoud (2011) draws three big lessons: The first lesson is that it is not enough simply to introduce integration as a limit of sums. If we want students to understand integration as a limit of Riemann sums, then they need experience working with these sums in contexts that lead them to appreciate the importance of this definition. The second lesson is that, despite our efforts to define integration as a limit of sums, the working definition of integration for most university students will continue to be antidifferentiation; this is deeply embedded from their high school experience with calculus. The third lesson draws on the historical development of the concepts of calculus in the 17th century. Integration and differentiation have two distinct conceptualizations, one geometric and the other dynamic. The historical lesson is to focus first on the dynamic understanding of the FTC—where the function to be integrated is viewed as a rate of change and the definite integral as an accumulator of this change—and then to use this to build the geometric realization of the theorem, where the integral is viewed as a sum of infinitesimals.

According to Winsløw (2022), the calculus taught at university is the result of a didactic transposition process (CHEVALLARD, 1991) of the calculus of the 18th century, complemented by some superficial, later theoretical developments. He claims that mathematical analysis provides clear examples of an increasing distance between scholarly knowledge (including the knowledge possessed and developed by lecturers) and the knowledge actually taught. This needs to be considered by researchers who set out to study the conditions and constraints associated with the teaching of calculus.

The research was conducted at a Norwegian university in the mathematics course Basic Calculus 1 (MA1101, n.d.) during the autumn of 2019, which had 217 students from various study programs. Six first-year students participated in the research. However, in this paper, we focus on the performance of one student enrolled in the bachelor's program in physics (BFY, n.d.). This student's performance is compared to that of the other five participants, although detailed analyses of their praxeological work are not included in this discussion.

We analyze the praxeology built up by the selected student related to the performance of a set of four integration tasks. The study of this praxeological work, carried out right after entrance to university, in fact reveals, *by duality*, the conditions and constraints to which the student was subjected and therefore the reality of the teaching he had received in upper secondary school. In focusing on early calculus, the paper addresses the concern of Thompson and Harel (2021), that students' preparation for calculus in a European context is largely unexamined.

The research question set out to answer is the following:

> *What is the praxeology constructed by a university mathematics student to perform a set of integration tasks, t_1 to t_4, and what are the tools available or missing from his pre-existing praxeological equipment, supporting or hindering the performance of these tasks?*

The research study is conducted within the framework of the anthropological theory of the didactic (ATD; CHEVALLARD, 2019, 2024). In the analysis of the construction of a praxeology done by the observed student, we use the model of didactic moments (CHEVALLARD, 1999).

THE INSTITUTIONAL CONTEXT

The set of tasks *Q* solved by the observed student (and the five others) was part of the topic "integration" in Basic Calculus 1.[1] The textbook used was *Calculus: A complete course* (ADAMS & ESSEX, 2018). The topics in the course were studied chronologically in the following order: limits, continuity, and series; differential calculus; inverse functions; integral calculus (with focus on analytic

[1] The letter *Q* is used to denote the set of tasks, rather than the possibly more natural capital *T*. In the ATD, a capital *T* is reserved for a "type of tasks," one of the four components of a *praxeology* (CHEVALLARD, 2019), which will be explained later. *Q* highlights the fact that at the foundation of a mathematical task, there is a mathematical question.

integration); and first order ordinary differential equations.

The teaching unit on integral calculus spanned weeks 8 through 12, covering a significant portion of the course. According to the course description, Basic Calculus 1 emphasizes rigor, with the fundamental theorem of calculus and its applications being central themes.

The didactic system

The didactic system studied, labeled $\mathcal{S}=(X,\ Y,\ Q)$, consisted of a student $X=\{x\}$ (called John henceforth) and a teacher $Y=\{y\}$. At the time of data collection, John was two weeks into his first semester of a bachelor's program in physics. Previously, John had completed the highest level mathematics courses, Mathematics R1 and R2, in upper secondary school, which covered differential and integral calculus, probability, and vector arithmetic and geometry, as per the 2006 Norwegian curriculum (DIRECTORATE FOR EDUCATION AND TRAINING, 2006). Opting for a more theoretical approach, John replaced the mandatory calculus course (Calculus 1, n.d.) with Basic Calculus 1 (MA1101, n.d.) and Basic Calculus 2 (MA1102, n.d.) from the mathematics bachelor's program (BMAT, n.d.), citing a preference for in-depth study. These decisions led John to undertake an additional 7.5 ECTS credits beyond the standard requirement for his physics program.

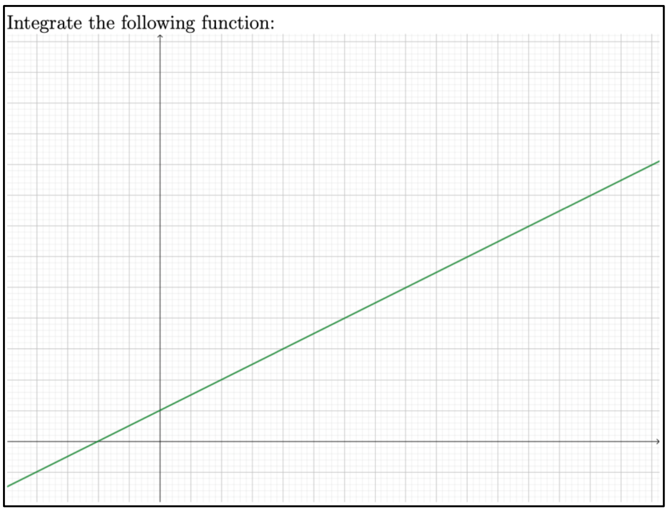
The teacher, y, who is the first author of this paper, was responsible for designing the mathematics tasks and served as the interviewer in the study. At the time of the data collection, he was a teacher assistant in the Basic Calculus 1 course and a PhD fellow at the department where the research was conducted. He held a master's degree in mathematics, specializing in numerical integration, and had completed pedagogical education. Additionally, y had six years of experience as a mathematics teacher, mainly at the upper secondary level.

The set of tasks Q, provided for study in $\mathcal{S}$, consisted of these four tasks:

t_1. Integrate the function $f(t) = t^2 + 2t$.

t_2. Integrate the function in Figure 1.

Figure 1 – Graph of function



Source: The authors

t_3. What can you say about $G'(x)$ in Figure 2?

Figure 2 – General function

Let $G(x)$ be defined as

$$G(x) = \int_{x-1}^{x+1} f(t)dt$$

What can you say about $G'(x)$?

Source: The authors

t_4. What can you say about $G'(x)$ in Figure 3?

Figure 3 – Periodic function

Let $G(x)$ be defined as

$$G(x) = \int_{x-1}^{x+1} f(t)dt$$

where $f(t)$ is a periodic function with period 2, so $f(t) = f(t + 2)$ for all $t \in \mathbb{R}$. What can you say about $G'(x)$?

Source: The authors

The material milieu consisted of 4 tasks, each printed on a separate A4 sheet of paper, in addition to blank sheets of paper. During the interviews, the students could use hand-held calculators. Use of the Internet or other resources was not mentioned, either by y or the students.

The purpose of Q

t_1 was meant as a warm-up task. t_2, t_3 and t_4 were formulated deliberately vague with the intention of examining how the students were able to tackle unfamiliar tasks related to integration. t_2 was meant to be similar to t_1 in the task formulation, but with differences in available information. We particularly wanted to see how they tackled constructing an antiderivative of a (linear) function, being given its graph and not its analytic expression. Here, the interpretation of a definite integral as an area is essential. Would they answer the task only based on the information they could get directly from the task, or would they make any extra assumptions, warranted or not, that would help them solve it by a method familiar to them? t_3 and t_4 were also similar, and they were designed particularly to test whether the students would be able to use the FTC in a more theoretical setting.

The FTC is a theme usually introduced in third year of upper secondary school, albeit usually not in a rigorous manner. However, connecting the integral and the antiderivative, and establishing a technique for calculating the area under a graph, is expected to be familiar to students when they enter the first calculus course at the university.

ANALYTIC TOOLS USED IN THE RESEARCH

Here we explain the tools we have used from the ATD, followed by a presentation of the reference model we developed for the set of tasks Q.

Didactic transposition and praxeology

The concept of *didactic transposition*, formulated by Chevallard (1991), explores how knowledge transforms as it moves from the institution where it is produced to the institution where it is taught. It addresses the changes that occur when scholarly knowledge is adapted into knowledge to be taught.

The scholarly knowledge selected for educational curricula undergoes significant simplification and restructuring to become accessible to students. These modifications are necessary due to the conditions and constraints of the educational institution, such as institutional needs, time constraints, students' prior knowledge and cognitive equipment.

Chevallard's (1991) framework outlines two stages of didactic transposition: external and internal. The external stage involves curriculum designers and policymakers selecting and transforming knowledge for educational purposes. The internal stage occurs within the classroom, where teachers further adapt the curriculum to their students' needs.

A critical outcome of didactic transposition is the creation of a *praxeological organization* of knowledge. A praxeology in the ATD is a unit composed of four components (CHEVALLARD, 2019): T, τ, θ, and Θ. The letter T denotes a type of tasks; τ stands for a technique (or a set of techniques) to solve the tasks; θ signifies a technology (i.e., a discourse) to describe and explain each technique; and Θ symbolizes a theory that justifies the technology. T and τ belong to the *praxis* block of a praxeology, whereas θ and Θ belong to the *logos* block. A praxeology $\wp$ is thus written $\wp = [T/\tau/\theta/\Theta]$, where $\wp$ is a model of the knowledge necessary to solve tasks of type T. These praxeologies form the building blocks of a praxeological organization, which encompasses the entire structure of the knowledge to be taught.

As illustrated by Strømskag and Chevallard (2024), didactic transposition processes lead to simplified praxeological organizations, usually resulting in *demathematization*, where scholarly knowledge is stripped of its rigorous theoretical underpinnings to be accessible for a broader range of students. To address the gaps created by didactic transposition, Strømskag and Chevallard (2024) propose the concept of an *archeorganization*, which is a meta-curricular framework designed to retain the core essence of scholarly knowledge while making it both accessible and viable for teaching. The purpose of an archeorganization for a mathematical topic is to bridge the educational divide between secondary and tertiary education regarding that specific topic.

Model of didactic moments

To analyze John's study process, we have used the *model of didactic moments* developed by Chevallard (1999). First, however, a remark on the use of this model is in place. Contrary to what is the case in most widespread didactic theories, where a binary distinction between teaching and learning is taken for granted, in the ATD, what is primary is the notion of *study* where teaching is only a possible means to learning (CHEVALLARD, 2024). In a book published in 1998, written by Chevallard, Bosch, and Gascón, aptly titled *Estudiar matemáticas. El eslabón perdido entre enseñanza y aprendizaje* [Studying mathematics: The missing link between teaching and learning], the authors write:

> In the case of school subjects, there is a tendency to confuse study activity with teaching or, at least, to consider only as important those moments of study when the student is in the classroom with a teacher. It is forgotten then that *learning*, understood as the *effect pursued* by study, does not occur only when the student is in the classroom with a teacher, nor does it occur only *during* teaching. The study—*or didactic process*—is a broader process which is not restricted to, but encompasses, the "teaching and learning process." (CHEVALLARD et al., 1998, p. 58, our translation)

Chevallard (2024) emphasizes that the model of didactic moments is a model of *study processes* in general, a tool to be used in analyzing the students' activity and in particular the students' role in the institutionalization of knowledge. Garcia et al. (2006), Artaud (2020), and Strømskag (2021) are examples of research studies where the model of didactic moments has been used this way. In what follows, we draw on Chevallard (1999, p. 249–255) in a brief presentation of the model of didactic moments.

The construction of a praxeology $\wp$ centered on a type of tasks T involves six "dimensions" called moments of the study process that generates the emergence of $\wp$; they are respectively: (1) the *moment of the first encounter* with the type of tasks T—when expressly organized, this first meeting may be part of a cultural-mimetic problem (e.g., meeting the problem by "playing" the mathematician, the physicist, etc.) or it may be part of a system of "fundamental situations" (in the sense of BROUSSEAU, 2002) or it may be a combination of these forms; (2) the *moment of the exploration of the type of tasks T and of the emergence of a technique* τ (of performing tasks t of the type T)—"it is indeed the development of techniques which is at the heart of mathematical activity" (CHEVALLARD 1999, p. 252, our translation); (3) the *moment of the constitution of the technological-theoretical environment* $[\theta/\Theta]$ relating to τ, when the *logos* part of the praxeology emerges; (4) the *moment of working on the praxeological organization* $\wp$ under development (in order to improve both the *praxis* block and the *logos* block)—this moment of testing the technique τ presupposes a corpus of adequate tasks; (5) the *moment of institutionalization* of $\wp$, the purpose of which is to specify the developed praxeology—that is, to define which elements will be integrated into $\wp$, and which will not; and (6) the *moment of the evaluation* (of both $\wp$ and of a person's relation to $\wp$), which is articulated at the moment of institutionalization (of which it is in some respects a sub-moment). Chevallard (1999) notes that the order of the different didactic moments is in fact largely arbitrary, because didactic moments are first of all a functional reality, before they become a chronological reality. Hence, the word moment in "didactic moment" has no temporal meaning.

A reference model for *Q*

To analyze the praxeology constructed by the observed student, we need a reference model that we present here. For t_1 through t_4, the question about what type of tasks they are, can be divided into several aspects. First, all the tasks are about concepts relating to the integral. Further, a connecting theme is the Fundamental Theorem of Calculus (FTC). More specifically, integration of a quadratic polynomial (t_1), a linear polynomial (t_2), a general function (t_3), and a periodic function (t_4). Second, we observe the way each task is formulated: t_1 and t_2 both include tasks of the type "Integrate the following function," whereas t_3 and t_4 are both of the type "What can you say about" a given function.

We expected that the solving of the first two tasks would be fairly algorithmic, centered on a set of techniques called integration, making these tasks suitable for exploring what the students understand this technique to be. The two most likely possibilities are the definite integral, and the indefinite integral, understood as the anti-derivative. The task t_2 was also designed specifically to induce a dissonance between these interpretations. As it is based on a graph instead of an algebraic expression, one could expect students to consider the possibility of interpreting the task as asking about an area, and thus also a definite integral. The last two tasks are more exploratory in nature, as they ask "what can be said about" the given functions. We expected that t_3 and t_4 would perhaps uncover more about techniques that involve no obvious ("correct") algorithms.

A third aspect is the different modes of representations featured in the tasks. Several descriptions of modes of representations exist (e.g., BRUNER, 1966; DUVAL, 2006). In this paper we distinguish between graphical and algebraic modes. We also acknowledge the importance of ordinary language, which is a linguistic mode used in task instructions and in explanations and arguments in solutions. For that reason, when talking about a "purely algebraic" or "purely graphical" mode, ordinary language is also involved. To illustrate this, in tasks t_1 and t_2, where the functions are respectively represented algebraically and graphically, the instruction "Integrate the function" uses ordinary language to direct the student's action on the mathematical object in question. As for tasks t_3 and t_4, the question "What can you say about the derivative?" requires the student to articulate their understanding of and reasoning about the mathematical object at stake, based on the graphical representations present in these tasks.

Reference solutions

We can solve the first task easily, using antiderivative rules for polynomial functions:

t_1: Apply the antiderivative to arrive at $\int f(t)dt = \int (t^2 + 2t)dt = \frac{1}{3}\,t^3 + t^2 + C$, where C is an arbitrary constant.

The second task can be solved in the same way, but not without making some extra assumptions about the graph. One such assumption is to say that the grid size is 1×1, in addition to the assumption of "what you see is what you get," that is, when the graph looks linear it is linear.

t_{2a}: Applying the point-slope formula, we get $f(x) = \frac{1}{2}\,x + 1$, and by taking the antiderivative, we have $\int f(x)dx = \frac{1}{4}\,x^2 + x + C$. We will call these solutions algebraic solutions to the tasks.

As noted, we had to make an assumption about the grid size in order to be able to find the antiderivative of this function. But would it be possible to give a solution to the task without making this extra assumption? In our opinion, this would be a more principled answer to the task, as it assumes that all information is already presented. But to do this, we need to consider what sort of information can be read out of the graph.

t_{2b}: First, we see that the graph is that of a linear function with a negative x-intercept and a positive y-intercept. If the straight line intersects the x-axis at the point $(x_0, 0)$ and the y-axis at the point $(0, y_0)$, its equation is $f(x) = \frac{y_0}{-x_0}(x - x_0)$. Suppose we do not know how to calculate an antiderivative of f. From the FTC (Part 1), we have that $\int_{x_0}^{x} f(t)dt$ is an antiderivative of f, which is equal to the area of the right-angled triangle whose vertices are $(x_0, 0)$, $(x, 0)$ and $(x, f(x))$. This area is equal to $\frac{1}{2}(x - x_0)\frac{y_0}{-x_0}(x - x_0) = \frac{y_0}{-2x_0}(x^2 - 2x_0x + x_0{}^2) = -\frac{y_0}{2x_0}x^2 + y_0x - \frac{1}{2}y_0x_0$, which is one of infinitely many antiderivatives of f. Hence, the indefinite integral is $\int f(x)dx = -\frac{y_0}{2x_0}x^2 + y_0x + C$, where C is any constant.

Also note that, whereas we did not change the mode of representation during the solving of t_1, we did so in t_2. It is possible to find solutions to t_2 using only geometric arguments, either by considering the different area elements, in a technique similar to Riemann integration, or by using slope fields (e.g., as presented by THOMAS & FINNEY, 1996). However, due to space limitation and of curricular reasons respectively, we will not present these techniques here.

t_3 and t_4: The tasks t_3 and t_4 are both quite flexible when considering what sort of statements can be accepted as solutions. Anything 'interesting' that "can be said about $G'(x)$" could be a solution. Particularly, since the only difference between t_3 and t_4 is that t_4 specifies that $f(t)$ is periodic, any solution to t_3 will also technically be a solution to t_4. However, since both tasks are based on describing a function, defined through a definite integral, using the FTC is reasonable.

t_3: From the FTC (Part 1), we have that, for a continuous function f on an interval $[a, b]$, the function $F(x) = \int_a^x f(t)dt$, defined on the same interval, is an antiderivative of f—that is, $F'(x) = f(x)$. We have $G(x) = \int_{x-1}^{x+1} f(t)dt = \int_{x-1}^{a} f(t)dt + \int_a^{x+1} f(t)dt = \int_a^{x+1} f(t)dt - \int_a^{x-1} f(t)dt = F(x+1) - F(x-1)$. Because of the linearity of the differentiation operator, we have $G'(x) = F'(x+1) - F'(x-1) = f(x+1) - f(x-1)$.

t_4: We know from t_3 that $G'(x) = f(x+1) - f(x-1)$. Given that $f(t) = f(t+2)$ for all t, we get $f(x-1) = f(x-1+2) = f(x+1)$. Hence $G'(x) = f(x+1) - f(x-1) = f(x+1) - f(x+1) = 0$. So, when $f(t)$ is periodic with period 2, we know that $G'(x)$ will always be equal to 0.

We see that for all four tasks, the technology θ of the techniques used is based on the FTC. For t_1 and t_2, we used that the processes of integration and differentiation are inverses of one another; for t_{2b} and t_3, it is essential that the integral of f with a variable bound of integration, $\int_a^x f(t)dt$, is an antiderivative of f. Further, for t_3, we used the additive property of definite integrals. t_4 is just a special case of t_3, where we used the periodicity of f.

METHODS USED

Data collection and case selected

The six students participating in the broader study were recruited in the beginning of the semester, where they could sign up voluntarily as participants. Before they agreed to take part, it was made clear that participation in the study would not influence their final grade in the course.

An adapted form of task-based interviews (MAHER & SIGLEY, 2020) was employed as data collection strategy. The students were interviewed twice while working individually to solve tasks on integration. The first interview was done before the introduction of integral calculus in the lectures, and the last one was done some time after integral calculus had been taught, but before the exam. The interviews, conducted in a small meeting room, were video-recorded and transcribed for subsequent analyses.

Both interviews were divided into three parts. The first part involved asking general questions about the student's background. In the second part, students were encouraged to verbalize their thinking while solving tasks presented in written form. After attempting to solve each task, the next one was introduced until all tasks were addressed. The interviewer only intervened to prompt the student to speak if they were silent for a prolonged period. The third part consisted of a more open discussion of the tasks.

The second interview was also divided into three parts. The first part focused on the student's perceived development during the course, while the last two parts mirrored the first interview. Interview 1 included tasks t_1 through t_4, whereas Interview 2 included tasks t_3, t_4, and a fifth task not discussed in this paper.

We selected John's praxeological work on Q as a focal case. This choice was influenced by his distinctive use of techniques and argumentation during the interviews, where he displayed a proficiency in tackling complex and unconventional problems. As mentioned in the introduction, the analysis of John's case is informed by the performances of five additional participants. While these contributions enhance our discussion, they are not detailed in this paper.

Methods of analysis

The chapter on integration in the textbook used by John in Grade 13 has been examined through a partial praxeological analysis, discerning the *praxis* and *logos* blocks of the chapter's exposition. Although presented first, the textbook analysis was conducted in response to our initial findings from the analysis of the student's praxeological work.

The analysis of John's praxeological work was conducted in two steps. The first step involved constructing a flowchart of the solving process for each task, including the different subtasks that emerged. This began with observing how John interpreted the task, noting all possible interpretations he mentioned. Subtasks were then identified and organized into a flowchart to display their dependency structure. The relative temporal ordering throughout the interview was represented by numbering the nodes in the flowchart. Only steps explicitly identified by John, such as expressing the need to perform a certain action or testing alternative solutions, were considered subtasks.

In the second step, the developed praxeology was examined, using the flowcharts to achieve three goals. First, the flowcharts mapped out the argument structure, which is crucial for examining the *logos* of the praxeology. For example, the structural part of a flowchart illuminated how a given technique was justified. Second, by identifying subtasks, potential connections to other praxeologies were revealed. Third, the flowcharts highlighted different didactic moments more clearly. For instance, the moment of the first encounter was characterized by how John understood the task's requirements, while the moment of constituting the technological-theoretical environment could be observed in the justifications of techniques. Additionally, the exploration of types of tasks was evident when there were multiple interpretations or when tentative answers were presented and validated.

The following analysis is based on transcripts of video-recordings of John's solutions to tasks t_1 through t_4.

INTEGRATION IN A TEXTBOOK ANALYZED

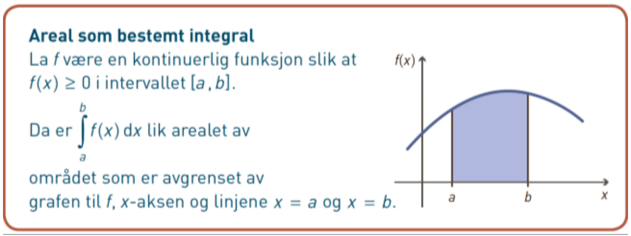
Here, we present a partial praxeological analysis of the main elements of integration as outlined in *Matematikk R2* (HEIR et al., 2016)[2], the textbook used in the mathematics course taken by John in Grade 13. This is relevant since the textbook brings to light an important aspect of the reality of the teaching John had received before entering the university—this informs about the conditions and constraints to which he was subjected during the praxeological work on Q.

Chapter 1 in the textbook is titled "Integration" and consists in three subchapters: "Definite Integral," "Indefinite Integral," and "Definite Integral by Antidifferentiation." In the first subchapter, the authors start by introducing *area* as a *definite integral*: Let f be a continuous function with $f(x) \geq 0$ on the interval $[a, b]$. Then $\int_a^b f(x)dx$ is equal to the area of the

[2] Figures and "boxes" with red frames taken from this textbook are reproduced with permission from the publisher, Aschehoug. All figures are made by Eirek Engmark at "Framnes Tekst & Bilde AS."

region bounded by the graph of *f*, the *x*-axis, and the lines $x = a$ and $x = b$ (see Figure 4).

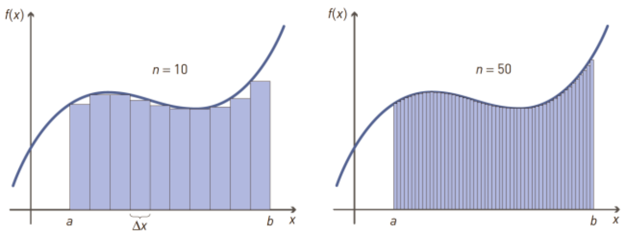
Figure 4 – Area as definite integral



Source: Heir et al. (2016)

This is followed by sections on: interpretation of the area under the graph of a function; the sign of the definite integral depending on the area being above or below the *x*-axis; and areas between two graphs. Then follows a more formal definition of the definite integral, that is, the *definite integral as the limit of a staircase sum*. The authors start with a continuous function *f* on the interval [*a*, *b*], where $f(x) \geq 0$. Figure 5 is used to illustrate two partitions ($n = 10$ and $n = 50$) of the region bounded by the graph of *f*, the *x*-axis, and the lines $x = a$ and $x = b$, where the width of the rectangles (i.e., the mesh width of the partition) is given by $\Delta x = \frac{b-a}{n}$.

Figure 5 – A region partitioned into rectangles using two different mesh widths



Source: Heir et al. (2016)

It is then explained that the area of the colored region is given by the sum of the areas of rectangles: $f(x_1) \cdot \Delta x + f(x_2) \cdot \Delta x + \cdots + f(x_n) \cdot \Delta x = \sum_1^n f(x_i) \cdot \Delta x$, where, for each rectangle, x_i is chosen so as to give the least value of $f(x_i)$. This sum is called a *lower staircase sum* for *f*. It is then argued that when the mesh width of the partition tends to zero, the staircase sum tends to the area of the region in question. Now, the definite integral of *f* from *a* to *b* is defined as the *limit* of the lower staircase sum as *n* tends to infinity, as symbolized in Figure 6.

Figure 6 – Definition of definite integral as the limit of a sum

Bestemt integral som grenseverdi

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x \text{ , der } \Delta x = \frac{b-a}{n}$$

Source: Heir et al. (2016)

The second subchapter, titled "Indefinite integral", starts with an example, pointing out that the derivative of x^2 is $2x$ and, further, that x^2 is called an *antiderivative* of $2x$. The authors go on to comment that, because the derivative of a constant is 0, the derivative of $x^2 + 5, x^2 - 8$ and $x^2 + \sqrt{3}$ is $2x$ as well. This observation is generalized by $(x^2 + C)' = 2x$, with $C \in \mathbb{R}$, after which they conclude that "$x^2 + C$ are all antiderivatives of $2x$". The operation that involves finding the function when knowing its derivative is introduced as *antidifferentiation*, and a generalization of the given example is recapitulated in Figure 7: "If $F'(x) = f(x)$, we say that *F* is an antiderivative of *f*. All antiderivatives of *f* are then given by $F(x) + C$."

Figure 7 – Definition of antiderivative

Hvis $F'(x) = f(x)$, sier vi at *F* er en antiderivert av *f*.
Alle antideriverte av *f* er da gitt ved $F(x) + C$.

Source: Heir et al. (2016)

Next, the authors notify that anti-differentiation of a function is usually called *integration* of the function, after which the notions of indefinite integral, integrand, and integration constant are introduced through the same example used above: $\int 2x dx = x^2 + C$. It is concluded that an *indefinite integral* is therefore all antiderivatives of the integrand, a definition symbolized thus (displayed in Figure 8): $\int f(x)dx = F(x) + C$, where $F'(x) = f(x)$ and $C \in \mathbb{R}$. Note the vagueness of the definition; it is not clear from the symbolic notation whether the indefinite integral is the *set* of antiderivatives or whether it is *any* of the antiderivatives. (After this follows a presentation and justification of integration rules.)

Figure 8 – Definition of an indefinite integral as all antiderivatives of the integral

Ubestemt integral

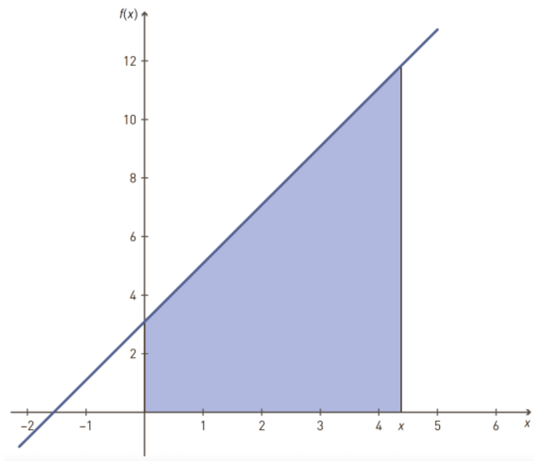
$\int f(x)\,dx = F(x) + C$, der $F'(x) = f(x)$ og $C \in \mathbb{R}$

Source: Heir et al. (2016)

We notice that what is displayed in Figure 7 can be seen as a *denaturation* resulting from the didactic transposition process that the notion of antiderivative in the textbook has undergone: the first part of the FTC (as presented in the introduction) has been transformed into a definition of an antiderivative without including the necessary condition about its nature—that the antiderivative is $\int_a^x f(t)dt$, a function referred to in some university textbooks as the *area function* (e.g., EDWARDS & PENNEY, 2002; BRIGGS & COCHRAN, 2011). In what follows, we will examine the consequences of this denaturation when it comes to the *logos* of calculating definite integrals by antidifferentiation. Before that, however, it can be noted that the very notion of indefinite integral is problematic also in university textbooks. The common notation for an indefinite integral of f with respect to x is $\int f(x)dx$, being defined as *all* antiderivatives of the function $f(x)$. In *A Comprehensive Textbook of Classical Mathematics*, written by Griffiths and Hilton (1970), the authors deal with the indefinite integral in quite an elaborated way (see Chap. 30, pp. 495–497). For instance, the indefinite integral of f is by definition the *set* of all primitives (i.e., antiderivatives) of f and is for historical reasons denoted by $\int f(t)dt$. They explain further that "the particular primitive G such that $G(a) = 0$ is called the *definite integral* of f and denoted by $G(x) = \int_a^x f(t)dt$ " (GRIFFITHS & HILTON, 1970, p. 495). Neuhauser (2011), for her part, explains that the symbolic notation $\int f(x)dx$ is a "convenient shorthand" for $\int_a^x f(t)dt + C$, where C is a constant (p. 299).

In the last subchapter on integration in *Matematikk R2*, titled "Definite Integral by Antidifferentiation," the authors start with a linear function, $f(x) = 2x + 3$, and compute the area between the graph of f and the x-axis between $x = 0$ and an arbitrary x-value greater than 0 (see Figure 9).

Figure 9 – The area of a trapezium with variable distance between parallel sides



Source: Heir et al. (2016)

The bounded region is described as a trapezium with parallel sides of lengths $f(0)$ and $f(x)$, separated by a distance x. Using the trapezium area formula, the area function F is derived as $F(x) = \frac{(f(0)+f(x))\cdot x}{2} = x^2 + 3$. Then an example calculates the area from $x = 2$ to the upper limit of $x = 5$. It is shown that the area A of the trapezium in this case is given by $A = F(5) - F(2) = 40 - 10 = 30$. This calculation ties back to the definition of area as a definite integral presented in the first subchapter, leading to the conclusion that the area A equals the definite integral of $f(x)$ from $x = 2$ to $x = 5$, which is equal to $F(5) - F(2)$.

There are no properties or notation implying that the area function F for the function f in question is $F(x) = \int_a^x f(t)dt$, for a continuous function f on a closed interval $[a, b]$. What the textbook, without further ado, is drawing on—that the area function is an antiderivative—is in fact Part 1 of the FTC presented in the introduction of this paper. This constitutes a missing link in the textbook between the notions of definite and indefinite integrals. That the area function is an antiderivative does not belong to the *logos* of the type of tasks which involves evaluating a definite integral by finding an antiderivative and evaluating the difference between the values of this antiderivative at the endpoints of the interval in question. The preceding elaboration of the particular trapezium is used as a generic example to introduce and explain the second part of the FTC: The authors ask the reader to notice that $F'(x) = f(x)$ and claim that this means that the area function F is an antiderivative of f. They assert further that this connection between the definite integral and an antiderivative, called the *Fundamental Theorem of Calculus*, turns out to apply generally. This result is presented symbolically, as shown in Figure 10. The authors do not provide a reference to, or mention the need for, any proof of the theorem. The presentation ends with a demonstration of the fact that the simplest antiderivative—the one without the constant C—can be used when evaluating definite integrals, since the constant will be cancelled anyway when the difference between the values at the endpoints of the interval is calculated.

Figure 10 – Technique for evaluation of a definite integral using an antiderivative

La F være en antiderivert av f slik at $F'(x) = f(x)$. Da er

$$\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a)$$

Source: Heir et al. (2016)

We have examined the treatment of integral calculus in two other Grade 13 textbooks for the same mathematics course, *Sigma R2* (SANDVOLD et al., 2015) and *Sinus R2* (OLDERVOLL et al., 2015). They have quite similar presentations as that in *Matematikk R2* analyzed here, and they have the same missing link between the definite integral and indefinite integral as demonstrated here. We will see how this deficiency likely plays a role in explaining John's struggle with t_2, where construction of an antiderivative from an area is central (see t_{2b} in "Reference solutions").

ANALYSIS OF JOHN'S PRAXEOLOGICAL WORK

John successfully arrived at the reference solution to t_1 and the least general solution to t_2 during the first interview, although he made an error with the coefficient in t_2 (Transition T7 → L7 in Figure 11). While interpreting the tasks as finding the indefinite integrals of the functions is valid, as outlined in the reference model, John also considered alternative interpretations. Specifically, when solving t_2, he proposed that the task could involve examining the area under the graph.

The technological-theoretical dimension

Solution to task t_1: Solving the first task proved to be no difficulty for John, as can be seen from the following excerpt.

> John: Yes, this one is so easy that for this one I can really just use a normal algorithm. So, then I would really just have written that the integral of $f(t)$, then, would be equal to…, I mean since it is inverse integration..., no inverse differentiation, this would be, eh… $\frac{1}{3}t^3 + t^2 + C$.

John's solution to task t_1 involved finding the antiderivative of a polynomial using a linear, algorithmic method, demonstrating a well-established praxeology centered on integrating polynomials. He recognized this method as the inverse of differentiation, linking it effectively to the praxeology of differentiation. John described the task as straightforward, indicating his proficiency with the technique, a pattern consistent with other students in the study.

Although the task alone does not fully explore all didactic moments, it showcases John's evaluative and explanatory understanding. To gain a comprehensive picture, it is essential to analyze how this solution technique compares to the approach taken in t_2, which has a similar structure.

An initial exploration of an unusual task

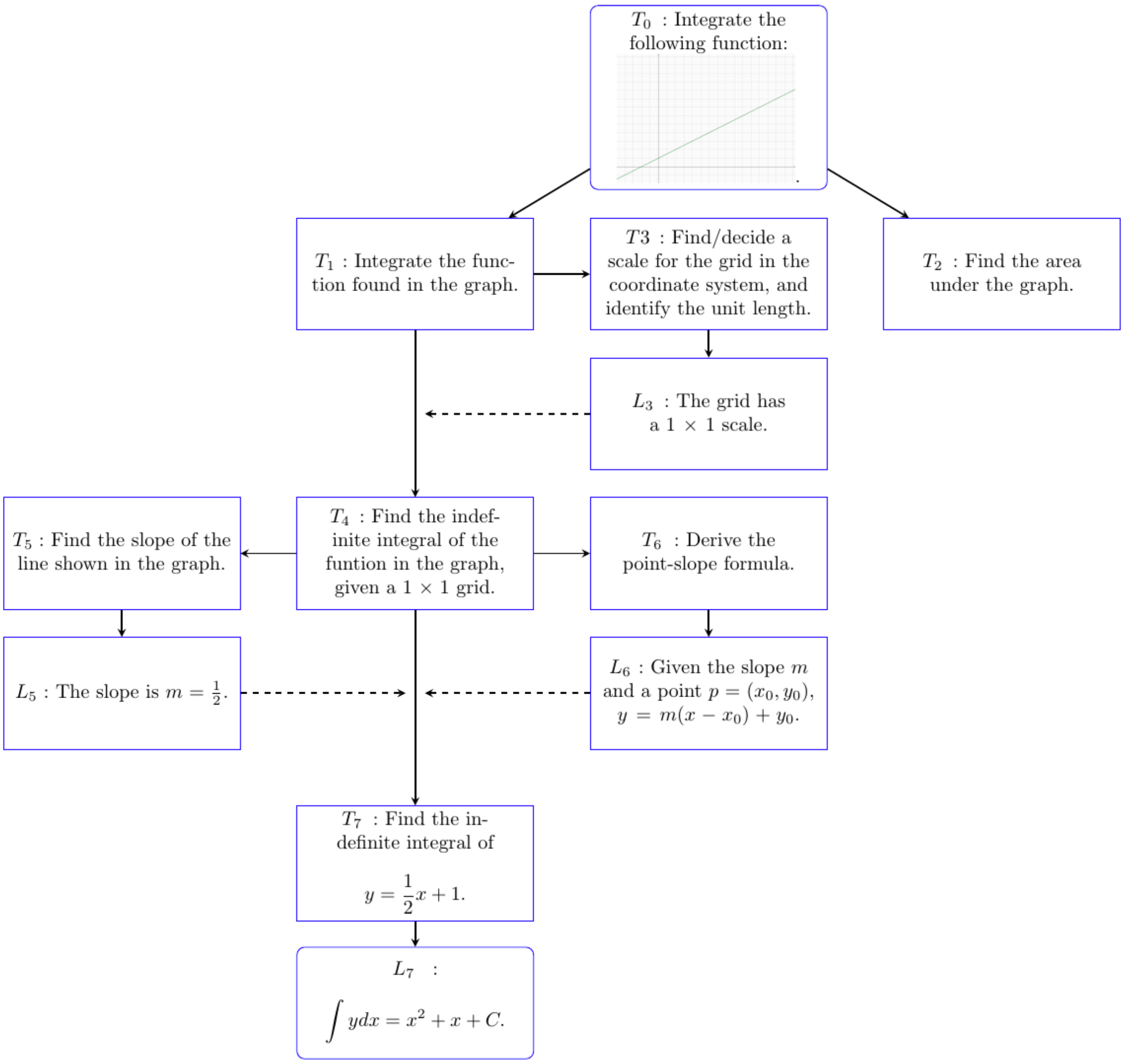
Task t_2 did not seem to pose a significant difficulty for John, though it required more effort than task t_1. He began by attempting to interpret the meaning of integrating the function (see Figure 11).

Here is the first major difference between t_1 and t_2. John mentions two ways of interpreting the task, compared to only one in t_1, and contrasts "assigning units" with thinking about it "more as an area":

> John: Here, the first that I think of is, eh…, it's like… the units here [points to the graph]. Should I assign units myself, or should I think about it more as an area?

These two alternatives go exactly along the lines suggested in the reference model. John did however not follow the area interpretation, but the episode clearly shows a conflict between the two interpretations, which was not present in t_1. The choice of interpretation of the concept of integration seems therefore related to the mode of representation used in the task. After deciding that integration also here means finding the antiderivative of the function, he prepares for an algebraic solution by changing the register from a graphical to an algebraic representation.

Figure 11 – John's solution to t_2



Source: The authors

Attempt to solve t_2

John started by assuming that the scale of the coordinate system is 1×1. Using the graph to find the inclination and y-intercept of the graphed line, he found an expression of the function. For the rest of the solving process, he used algebraic techniques. Notably, John did not remember the point-slope formula for linear functions, but redeveloped it by remembering that the inclination can be written as $m = \frac{y_1 - y_0}{x_1 - x_0}$, and rewrote it on the standard form $y = m(x - x_0) + y_0$, where y_1 and x_1 are renamed to y and x. Using this, he found the expression $y = \frac{1}{2}x + 1$ that he integrated and then arrived at the (incorrect) solution $\int ydx = x^2 + x + C$ instead of $\int ydx = \frac{1}{4}x^2 + x + C$, perhaps because he read $2x$ instead of $\frac{1}{2}x$. John then considered whether he could find a value for C, but quite quickly saw that it was not possible, something which also points at a connection between the representation he used and the interpretation of the task:

> John: And you see here at once that C must be… no, actually you don't. You need initial values for that, so… no…

> That would be my answer to that task. But it… it is a very loose task since you don't have anything really defined… what, how big… and so on. But I guess it will… so the problem here is that… let's say that this here is one half [he points to the point on the *y*-axis previously marked with 1], and this one here is one [he points to y = 2], then you would have a completely different slope. And you would get a completely different result then. You can also, of course, calculate the area under the graph. But that will be a *definite integral* [our emphasis]. So, it depends on what you are looking for. Well, I think I'm done with it then.

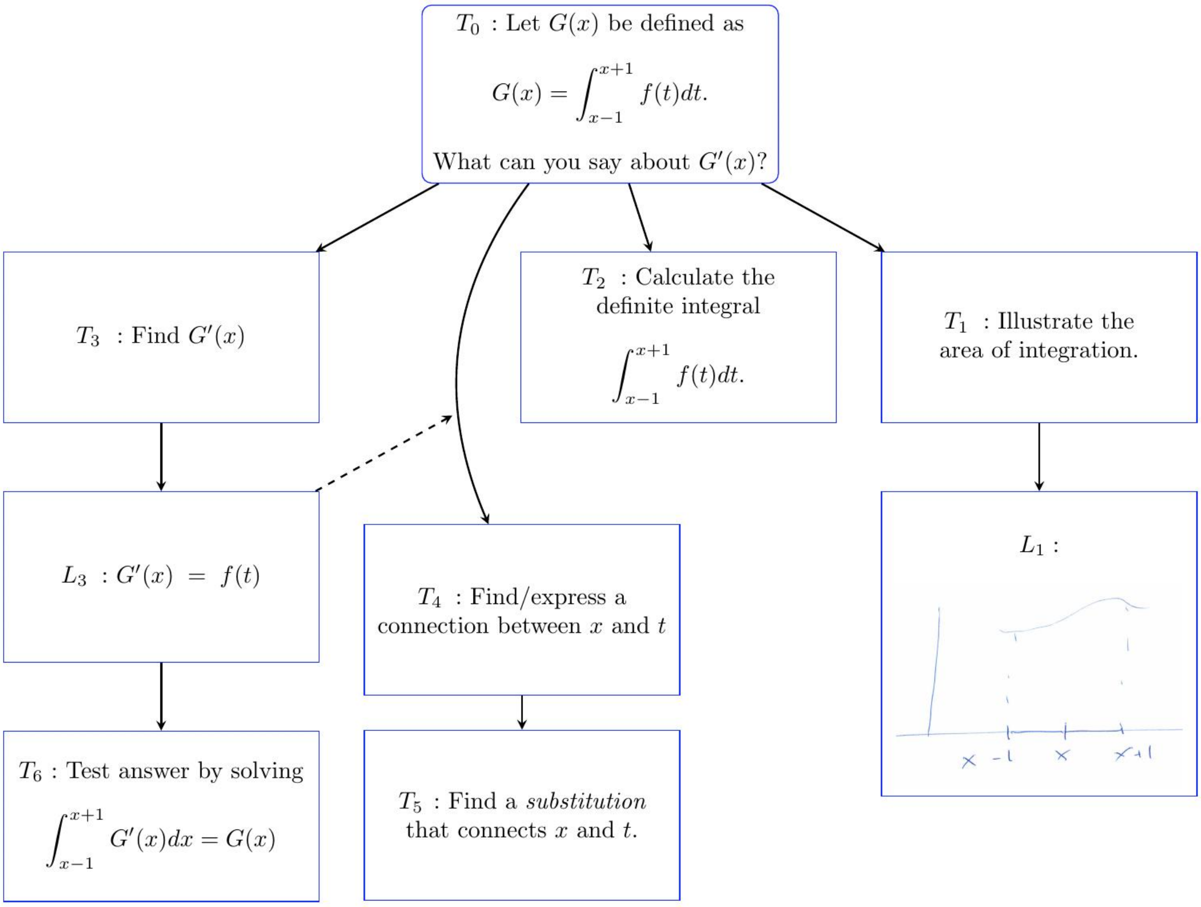
In t_2, John experiences a first encounter with the problem. Although he had dealt with graphical representations of functions previously, this specific task was new to him. His initial uncertainty and exploration of two potential interpretations, pondering whether the task involved calculating an area or determining an indefinite integral, illustrate his unfamiliarity. He noted that his approach "depends on what [the teacher] is asking about," signaling his attempt to discern the rules of the task. While John had to rediscover the technique for algebraically expressing the function from the graph, finding the antiderivative was comparatively straight-forward. This points to an emerging technique of algebraically representing a function from a graph prior to integrating, which could help in bridging the techniques developed in t_1 and t_2 and in forming a local praxeology out of the two point-praxeologies centered on these tasks.

When we examine t_1 and t_2 together, we also see a few patterns emerging. First, there was a strong preference for interpreting t_2 as being about an indefinite integral, similarly to t_1, instead of a definite integral. Further, John interpreted the task as being more specific, rather than more general, particularly as he followed the reference solution to t_{2a}, rather than the one to t_{2b}. In addition, although both graphical and algebraic techniques were attempted, John seemed unable to use the graph directly in exploring t_2. What John declared in the excerpt from the transcript above, suggests that a definite integral measures an area, and its value depends on the units chosen, but, according to John, the calculation of an area would not be suitable because it would only ever give us the value of a definite integral even though we were trying to determine an "indefinite integral." What John seems to be saying in this case is (partially) wrong, a claim we substantiate by the reference solution to t_{2b}. His erroneous inference is likely caused by the missing link, $\int_a^x f(t)dt$, between the definite integral and the indefinite integral, which we identified in the textbook he had used in Grade 13. In John's conception of integration, area is a number, not a function. This might explain his praxeological shortcoming, as suggested by the utterances quoted above.

Attempt to solve t_3

While solving t_3, John did start with a graphical technique when illustrating the integration area on the abscissa axis in a coordinate system (see L1 in Figure 12).

Figure 12 – John's solution to t_3



Source: The authors

But the way John seemed to interpret the task, to calculate the definite integral $\int_{x-1}^{x+1} f(t)dt$, indicates a more algebraic approach. John continued by focusing on the task of connecting the derivative and the integral:

> John: Eh… the derivative of $G(x)$ will be, I mean … Now, wait a bit. I have to think a bit here. If the *integral* of f is G, then the *derivative* of G must be f… of t. The problematic part here is that this is x, and then this is t. I didn't notice that until now. Those are two different variables.

This shows first a refocusing on algebraic aspects of the task: the focus on the presence of two different variables in the two functions, and the focus on the connection between integrals and derivatives as being *somehow* opposites of each other, although the exact nature of this "oppositeness" is unclear. It seems here that John has not been exposed to the difference between *bound* and *free* (or "dummy") variables. Once again, the observation made reveals aspects of the teaching provided in upper secondary school. But it also shows the point where John discovered that his initial approach might have some limitations. This prompted an exploration of what different alternatives he had. The result from the FTC, that calculating an integral is the opposite of calculating a derivative, is dependent upon the integral involved being an *antiderivative*, while in t_3 the integral is a *definite* integral with variable limits of integration. This seems to be what was troubling John. After having previously suggested that $G'(x) = f(t)$, he continued to examine the consequences of that idea:

> John: Wait a bit. The only thing that will happen is that [Inaudible] If I write like this, then, that $G(x)$ ... [Writes down $G(x) = \int_{x-1}^{x+1} G'(x)$] ... and that must be equal to… No, or yes. It must be equal to something like this then. G ... of dx [appends dx to the expression that he just wrote.] Or I don't know if this is correct. [Crosses over the integration limits.] This has to be an indefinite integral then.

What John attempts, and more importantly what he does not attempt, suggests that the teaching he had received did not emphasize much the *additivity* of definite integrals, which we used in the reference solution to t_3. John did not arrive at the reference solution, although he was able to see that the result he tried to use did not apply to definite integrals but to antiderivatives.

Attempt to solve t_4

The task t_4 introduced the extra constraint that $f(t)$ is supposed to be 2-periodic. In the beginning, John started by recalling what a periodic function is. He ended up by presenting the archetypical examples of sine and cosine functions. After this, he tried to use the additivity of definite integrals in rewriting the integral as a sum of two separate integrals, but the expression he arrived at, $G(x) = \int_x^1 f(t)dt + \int_{-1}^x f(t)dt$, was not correct. (A correct formulation would be $\int_{x-1}^a f(t)dt + \int_a^{x+1} f(t)dt$; refer to the section "Reference solutions") This is a very telling mistake. Second, it also shows that, although he had some familiarity with the FTC from upper secondary school, he had trouble seeing how it could be applied in a new situation. Having variable integration limits seems to have made it difficult for him to interpret the integration area correctly. He did, however, manage to identify that this was *at the heart* of the challenge. He saw that there were two variables in the expression, and thereby identified the need for a technique which he tried to discover. But as seen in his examination of t_3, John gave up on the task at this point.

One of the things John *did* manage to say about the problem, was that the integral of a periodic function would still be a periodic function. His explanation was based on his prior knowledge of derivatives. It is not rigorous, however, and mainly based on a paradigmatic example: It is therefore a mathematical hypothesis based on the examples known to him. This is part of the praxeological work, even if it does not constitute a proof.

> John: The function is periodic, and had there been an x there, then one could have said that the derivative is also periodic. Right? If it were a sine function, for example. That one is what we call periodic, I guess. And then the derivative of sine will be cosine, of course, which is shifted.

Comparison between Interviews 1 and 2

In the second interview, John was presented with t_3 and t_4 again. It is not feasible to present a detailed analysis of this interview, but some comparison to how he solved t_3 and t_4 is warranted. In this interview, he managed to solve task t_3 but not t_4. While solving t_3, he arrived at the correct conclusion that $G'(x) = f(x+1) - f(x-1)$. The key technique he used was to define a new function $F(t)$, such that $F'(t) = f(t)$. This allowed him to apply the FTC to arrive at the partial answer $G(x) = F(x+1) - F(x-1)$. He further identified this technique of defining a new *general* function, and working on that, "as if it was a specific function", to be the key to why he was able to solve the task this time.

John also applied the chain rule when differentiating the function $G(x)$. Although being redundant in this case, it is not a mistake, and would be necessary if the limits of integration were different. But it does give some hints about the developing praxeology. Although the techniques seemed to be known to John, and he had gained some experience, compared to the first interview, the knowledge about when certain parts of the technique are necessary or not, was lacking. So, whereas John was in the process of exploring the type of tasks presented in the first interview, he now seemed to be working on the praxeological organization, and on the institutionalization. He had the techniques at his disposal, and could reasonably use them to solve the task, but the *efficient* use of them was not yet fully developed. His own explanation why he managed to solve the task in the second interview, was that he had more experience in defining unknown and general functions to examine problems like this. This indicates that John now seemed to see the concept of function more as a mathematical object, instead of the algorithmic view seen in the first interview.

John's techniques

First, we see a strong prevalence of algebraic techniques in all four tasks, indicating that when John uses graphs and graphical techniques, he seems to do so mainly for illustrative purposes. In addition to the graph that was part of t_2, he only made two other graphical illustrations, both as part of solving t_3. Notably, he did not illustrate the periodic function asked for in t_4.

Second, the connection between the derivative and the integral seems particularly strong, and especially as an explanatory factor in reasoning about why the techniques John applies work. This is the case both in t_1, where he uses it to explain why the algorithm of integrating a polynomial works, and in t_3, where it leads to the erroneous first attempt at a conclusion that $G'(x) = f(t)$. This leads us to believe that the important connection between the derivative and the integral is a complex part of the technological-theoretical link in a praxeology of integration. Understanding the precise nature of the integral, the antiderivative and the derivative is a crucial step in building up the *logos* block.

Insights from other students' performances

We can see many of the same tendencies also among five other students, and here we just present a short summary of observations from their first interview round. Task t_1 was solved rather quickly by all, using a single technique of integrating polynomials.

Task t_2 had considerably more variation in solution techniques. Although the presence of a graph made some of them consider finding the area under the graph, they all used algebraic techniques to arrive at the reference solution to t_{2a} as their final answer.

Both t_3 and t_4 posed significant challenges for the students. In the case of t_3, John was the only one who managed to find the solution $G'(x) = f(x+1) - f(x-1)$. The other five students concluded erroneously that $G'(x) = f(t)$. With respect to t_4, four of the other five students failed to arrive at a solution, whereas the fifth student managed to solve it through a graphical exploration, concluding that $G'(x) = 0$ (see TOPPHOL, 2021).

DISCUSSION

The treatment of the FTC in the three Norwegian textbooks, which we interpret as the result of *didactic transposition* processes (STRØMSKAG & CHEVALLARD, 2024), is not an isolated phenomenon. A similar approach can be found in the German textbook titled *Lambacher Schweizer* (BRANDT et al., 2015), which has been the leading mathematics textbook series for the upper secondary level (Gymnasium) in Germany for decades, according to Stark (2011). For instance, in this textbook, the area function (referred to as "Integralfunktion") is introduced *after* the treatment of the FTC, where it is stated without proof that this function is an antiderivative. Consequently, the area function is not explicitly provided as the link between antiderivatives and definite integrals, similar to the approach seen in the Norwegian textbooks. Despite its strong reputation for clarity and comprehensiveness (see "LAMBACHER SCHWEIZER," 2011), we find this characteristic in the German textbook.

The praxeology built up by John around the set of tasks Q, is mainly based on algebraic or symbol-based techniques. During the first interview, the most common technique he used was calculating the antiderivative. Finding algebraic expressions from graphical representations (in t_2) and representing classes of functions graphically (in t_3 and t_4) were also common themes. But these techniques were still strongly based on algebraic procedures. Manipulation of the graphical representations and arguing based on them made up a very limited part of the praxeology constructed by John. The development from Interview 1 to Interview 2 was also mainly algebraic. The most important new technique was to define *general* functions and to do calculations on them. It is likely that the notion of "rigor" put forward by the teaching institution has strongly induced the privileged use of algebraic procedures and the minimization of graphic considerations, as reflected in the praxeological work of John (and similarly observed in the five others).

Furthermore, the *logos* appears to be relatively underdeveloped, especially concerning integration techniques. This aligns with John's description of upper secondary mathematics as being fairly algorithmic. When he did provide explanations beyond algebraic calculations, they were often limited to recalling facts (e.g., explaining integration as "inverse differentiation") or arguing by examples (e.g., using the sine function to illustrate a general periodic function). The exception to this was when identifying difficulties; for instance, when he needed to establish a connection between the variables x and t to solve task t_3. In such cases, he could accurately identify the challenge and explain why he was unable to solve the problem. However, he lacked the theoretical knowledge to make this connection until he developed the technique of defining a new function himself, a method he mentioned he had not previously encountered in any task.

Identification of an underdeveloped *logos* block is consistent with findings by Radmehr and Drake (2019), who found that students' conceptual skills of integral-area relationships were considerably less developed than their procedural skills (see also MAHIR, 2009). In the ATD, "conceptual skills" are part of the *logos* block whereas "procedural skills" are part of the *praxis* block. Our study adds to the mentioned studies in that the *logos* block, with its technological and theoretical elements, provides more detailed description of the conceptual part of the knowledge at stake—here, integral calculus. Furthermore, by considering the treatment of integration in the Grade 13 textbook used by John, this study points at the praxeological equipment of integration he likely developed at upper secondary level, and consequently, reveals conditions and constraints under which the praxeological work analyzed here was carried out. This enabled us to identify a potential causal relationship between the textbook's treatment of integration and John's praxeological shortcoming: an essential element of the FTC—area in terms of a function—was missing in the *logos* block of the textbook, which made tasks of type t_2 unsolvable. Since this flaw was observed also in other textbooks (Norwegian and German), our study has significance for a larger population.

From the analysis of John's praxeological work, a few key ideas emerge. First, we see that there is a focus on algebraic techniques, rather than graphical ones, a pattern corroborated by our observation of five other students. Although there were attempts to illustrate the problem graphically, most of the students failed at using the graphs in solving the problems, and they resorted to algebraic manipulations. A possible explanation concerns the emphasis on calculation in upper secondary mathematics: the examined textbooks tend to promote reliance on algebraic techniques, even during the exploration phase. This may even be strengthened at university, where they will encounter a focus on rigor.

A second idea is what we perceive as a challenge in the treatment of *parameters*. John (as well as the other students observed) is clearly used to handling polynomial functions in their algebraic form, and to some extent also trigonometric functions. But challenges emerge in two situations. First, relating to t_2, where there was a need to introduce the scaling of the graph itself and identify the unit length. Even if they managed to solve the task, several expressed discomfort with how the task was formulated; most of them failed to consider the more general solution (see reference solution to t_{2b}) and preferred to solve it as if it was a specific function (refer to reference solution to t_{2a}). Second, in tasks t_3 and t_4, the link between the two variables t and x proved to be challenging. Only by introducing a new helper function, $F(t)$, John managed to solve t_3 in the second interview. These observations highlight the following difficulties: regarding task t_2, the challenge lies in the conceptualization of an aspect of functions, such as the area under a graph, as itself a function; and concerning tasks t_3 and t_4, the challenge is in the generalization.

To conclude, John's praxeological equipment was predominantly composed of nonmaterial tools such as algebraic expressions and formulas, along with their manipulations. These tools are supplemented only sparingly by graphical representations, which are utilized mainly for illustrative purposes. However, a significant gap in his praxeological equipment was the underdevelopment of the *logos* block relating to the FTC, particularly evident in the integration techniques and conceptual explanations, which were often limited to simple recollections or example-based arguments. This deficiency is partly attributed to the didactic transposition evident in the textbooks used at secondary level, which tends to marginalize the conceptual linkage between antiderivatives and definite integrals, as observed in both Norwegian and German textbooks.

CONCLUSION

The use of graphs and figures in exploring mathematical problems has a long history (see e.g., EDWARDS, 1979; KLINE, 1972). Combining geometry and algebra was crucial in developing functional analysis and calculus, suggesting benefits in using both geometrical and algebraic techniques during exploration. The absence of these techniques among the interviewed students indicates a lack of previous exposure to such explorations, so a practical consequence of this study is the need for teacher education to address this imbalance by emphasizing the use of both graphical and algebraic methods.

In particular, when addressing the type of tasks that involve computing a definite integral of a continuous function f over a closed interval $[a, b]$ by finding an antiderivative and evaluating the difference between the values of this antiderivative at the endpoints of the interval, it is essential to clearly articulate in the *logos* block that the area function F for f, defined by $F(x) = \int_a^x f(t)dt$, serves as an antiderivative of f. This principle is essential for evaluating the definite integral using the FTC, which states that the integral from a to b of f can be computed by finding the difference $F(b)–F(a)$.

Moving beyond specific tasks, the overall examination of the *logos* block of integral calculus related to Q reveals similar deficiencies across the larger sample of students. These students predominantly relied on ritualistic approaches, often resorting to memorization rather than engaging in mathematical reasoning and justification when solving the four tasks.

This does confirm what has been shown already about students' understanding of integral calculus (e.g., RADMEHR & DRAKE, 2019; THOMPSON & HAREL, 2021). However, our study specifically connects these findings to a student's study situation where the impact of educational resources is considered. Through textbook analysis, we have identified a potential causal link between a deficiency in the observed student's praxeological equipment and the treatment of integral calculus in the textbook he had used at secondary level.

The result about the deficient *logos* block calls for a critical review of the resources used in mathematics education, highlighting the broader consequences of didactic transposition processes and the related institutional fragmentation, particularly in mathematics textbooks. Consequently, there is a need for heightened awareness among educators at both secondary and tertiary levels about the effects of didactic transposition. This emphasizes the necessity of addressing these influences systematically.

A practical implication of these findings in the field of teacher education is the need for educators and student teachers to be trained to understand the processes of didactic transposition and their effects on the mathematical knowledge to be taught. By becoming more familiar with didactic transposition analysis and its essential tool, praxeological analysis, along with principles like epistemic integrity, they will be more adept at evaluating and adapting their educational resources to more closely align with scholarly knowledge.

Ultimately, a strategic solution would be the development of an archeorganization for integral calculus, akin to the one created by Strømskag and Chevallard (2024) for the concept of concavity of functions. Such a framework, as a praxeological organization of integral calculus, would serve as a vital epistemic tool, guiding curriculum designers and textbook authors through didactic transposition processes specific to integral calculus. Teacher education institutions are particularly suited to develop and implement archeorganizations, as they play a critical role in preparing future educators. By understanding didactic transposition, educators and curriculum designers can better navigate the complexities of knowledge transformation, contributing to the development of praxeological organizations that ensure epistemic fidelity in mathematics education.

An essential future research step is the development of an archeorganization for integral calculus that retains the core essence of the scholarly knowledge while making it accessible and viable for teaching. Once developed, further studies should explore the long-term impacts of such a praxeological organization on mathematics curricula, textbooks, and students' comprehension of integral calculus.

REFERENCES

ADAMS, R.; ESSEX, C. **Calculus: A complete course.** 9th ed. Don Mills: Pearson, 2018.

ARTAUD, M. Praxeologies to be taught and praxeologies for teaching: A delicate frontier. In M. BOSCH et al. (Eds.), **Working with the anthropological theory of the didactic in mathematics education: A comprehensive casebook**. London: Routledge, p. 33–40, 2020.

BFY. **Bachelor's program in physics**, n.d. https://www.ntnu.edu/studies/bfy

BMAT. **Bachelor's program in mathematical sciences**, n.d. https://www.ntnu.edu/studies/bmat

BRANDT, D.; JÖRGENS, T.; JÜRGENSEN-ENGL, T.; RIEMER, W.; SCHMITT-HARTMANN, R.; SONNTAG, R.; SPIELMANS, H. **Lambacher Schweizer Mathematik Qualifikations-Phase. Leistungskurs/Grundkurs Nordrhein-Westfalen**. Stuttgart: Klett, 2015.

BRESSOUD, D. M. Historical reflections on teaching the Fundamental Theorem of Integral Calculus. **American Mathematical Monthly**, v. 118, n. 2, p. 99–115, 2011. https://doi.org/10.4169/amer.math.monthly.118.02.099

BRIGGS, W. L.; COCHRAN, L. **Calculus**. Boston: Pearson, 2011.

BROUSSEAU, G. **The theory of didactic situations in mathematics: Didactique des mathématiques**: 1970–1990 (N. Balacheff et al., Eds. & Transl.). Dordrecht: Kluwer, 2002. https://doi.org/10.1007/0-306-47211-2

BRUNER, J. S. **Towards a theory of instruction**. New York: Norton, 1966.

CALCULUS 1. **Basic calculus course in master's programs in technology**, n.d. https://www.ntnu.edu/studies/courses/TMA4100/2019#tab=omEmnet.

CHEVALLARD, Y. L'analyse des pratiques enseignantes en théorie anthropologique du didactique. **Recherches en Didactique des Mathématiques**, v. 19, n. 2, p. 221–266, 1999.

CHEVALLARD, Y. Introducing the Anthropological Theory of the Didactic: An attempt at a principled approach. **Hiroshima Journal of Mathematics Education**, v. 12, p. 71–114, 2019. https://www.jstage.jst.go.jp/article/hjme/12/0/12_1205/_pdf

CHEVALLARD, Y. What is questioning the world? Towards an epistemological and curricular break. In I. Florensa et al. (Eds.), **Extended abstracts 2022: Proceedings of the 7th International Conference on the Anthropological Theory of the Didactic**. Cham: Birkhäuser, 2024. https://doi.org/10.1007/978-3-031-55939-6

CHEVALLARD, Y. (with JOHSUA, M.-A.) **La transposition didactique. Du savoir savant au savoir enseigné**. 2nd ed. Paris: La pensée sauvages, 1991.

CHEVALLARD, Y.; BOSCH, M.; GASCÓN, J.

Estudiar matemáticas. El eslabón perdido entre enseñanza y aprendizaje. Barcelona: Horsori, 1998. https://diposit.ub.edu/dspace/bitstream/2445/174473/1/0037_970-18-1739-7_Matmaticas_Chevallard.pdf

DIRECTORATE FOR EDUCATION AND TRAINING. **Curriculum in mathematics for natural sciences specialization**, 2006. https://www.udir.no/kl06/MAT3-01?lplang=http://data.udir.no/kl06/eng

DUVAL, R. A cognitive analysis of problems of comprehension in a learning of mathematics. **Educational Studies in Mathematics**, v. 61, n. 1–2, p. 103–131, 2006. https://doi.org/10.1007/s10649-006-0400-z

EDWARDS, C. H. **The historical development of the calculus**. New York: Springer-Verlag, 1979.

EDWARDS, C. H.; PENNEY, D. E. **Calculus**. 6th ed. Boston: Pearson, 2002.

GARCIA, F. J.; PÉREZ, J. G.; HIGUERAS, L. R.; BOSCH, M. Mathematical modelling as a tool for the connection of school mathematics. **ZDM Mathematics Education**, v. 38, n. 3, p. 226–246, 2006.

GRABINER, J. Who gave you the epsilon? Cauchy and the origins of rigorous calculus. **The American Mathematical Monthly**, v. 90, n. 3, p. 185–194, 1983.

GRIFFITHS, H. B.; HILTON, J. P. **A Comprehensive textbook of classical mathematics: A contemporary Interpretation**. New York: Springer, 1970.

HEIR, O.; BORGE, I. C.; ENGESETH, J.; HAUG, H.; MOE, H. **Matematikk R2**. 2nd ed. Oslo: Aschehoug, 2016.

KLINE, M. **Mathematical thought from ancient to modern times**. New York: Oxford University Press, 1972.

LAMBACHER SCHWEIZER. In ***Wikipedia***, November 7, 2021. https://de.wikipedia.org/w/index.php?title=Lambacher_Schweizer&oldid=217074422

MA1101. **Basic Calculus 1**, n.d. https://www.ntnu.edu/studies/courses/MA1101/2019#tab=omEmnet.

MA1102. **Basic Calculus 2**, n.d. https://www.ntnu.edu/studies/courses/MA1102/2019#tab=omEmnet.

MAHER, C. A.; SIGLEY, R. Task-based interviews in mathematics education. In S. LERMAN (Ed.), **Encyclopedia of Mathematics Education**. Cham: Springer, p. 579–582, 2020. https://doi.org/10.1007/978-3-030-15789-0_147

MAHIR, N. Conceptual and procedural performance of undergraduate students in integration. **International Journal of Mathematical Education in Science and Technology**, v. 40, n. 2, p. 201–211, 2009.

NEUHAUSER, C. **Calculus for biology and medicine**. 3rd ed. Boston: Pearson, 2011.

OLDERVOLL, T.; ORSKAUG, O.; VAAJE, A.; HALS, S. **Sinus R2**. 2nd ed. Oslo: Cappelen Damm, 2015.

RADMEHR, F. **Exploring students' learning of integral calculus using revised Bloom's taxonomy**. Thesis (Doctorate in mathematics education) – Victoria University of Wellington, New Zealand, 2016.

RADMEHR, F.; DRAKE, M. Students' mathematical performance, metacognitive experiences and metacognitive skills in relation to integral-area relationships. **Teaching Mathematics and Its**

Applications, v. 38, n. 2, p. 85–106, 2019.

SANDVOLD, K. E.; ØGRIM, S.; BAKKEN, T.; PETTERSEN, B.; SKRINDO, K.; THORSTENSEN, A.; THORSTENSEN, R. **Sigma R2**, 2nd ed. Oslo: Gyldendal, 2015.

STARK, J. **65 Jahre Lambacher Schweizer: Ein Klassiker – Immer auf der Höhe der Zeit**. Stuttgart: Klett, 2011. https://www.klett.de/inhalt/media_fast_path/145/Festschrift_LS65_Internet.pdf

STRØMSKAG, H. Prospective teachers solving a percentage problem: An analysis of the construction of a praxeology. In B. BARQUERO et al. (Eds.), **Extended abstracts spring 2019. Trends in mathematics 13**. Cham: Birkhäuser, p. 77–86, 2021. https://doi.org/10.1007/978-3-030-76413-5_9

STRØMSKAG, H.; CHEVALLARD, Y. Didactic transposition and the knowledge to be taught: Towards an archeorganisation for concave/convex functions. **International Journal of Mathematical Education in Science and Technology.** Advance online publication, 2024. https://doi.org/10.1080/0020739X.2024.2305879

THOMAS, G. B.; FINNEY, R. L. **Calculus and analytic geometry**. 9th ed. Reading: Addison-Wesley, 1996.

THOMPSON, P. W.; HAREL, G. Ideas foundational to calculus learning and their links to students' difficulties. **ZDM Mathematics Education**, v. 53, n. 3, p. 507–519, 2021. https://doi.org/10.1007/s11858-021-01270-1

THOMPSON, P. W.; SILVERMAN, J. The concept of accumulation in calculus. In M. P. CARLSON; C. RASMUSSEN (Eds.), **Making the connection: Research and teaching in undergraduate mathematics**. Washington: Mathematical Association of America, p. 43–52, 2008.

TOPPHOL, V. A novel application of the instrumental approach in research on mathematical tasks. In G. A. NORTVEDT et al. (Eds.), **Preceedings of NORMA 20: The 9th Nordic Conference on Mathematics Education**. Gothenburg: NCM & SMDF, p. 265–272, 2021.

WINSLØW, C. Mathematical analysis at university. In Y. CHEVALLARD et al. (Eds.), **Advances in the Anthropological Theory of the Didactic**. Cham: Birkhäuser, p. 295–305, 2022. https://doi.org/10.1007/978-3-030-76791-4_23