

CREATIVITY, TACIT KNOWLEDGE AND MATHEMATICS EDUCATION

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“It were the mathematicians Thales, Pythagoras and Plato, who created metaphysics and metaphysics has always been the ape of mathematics. Seeing how the propositions of geometry flow demonstratively from a few postulates, men got the notion that the same must be true in philosophy” (Peirce, CP 1.130).

Abstract

Explicit theoretical knowledge forms a reality *sui generis*, which cannot be easily related to the dynamics of the real world. Teaching and learning depends therefore much on implicit or tacit knowledge and personal contact. In particular, mathematical proofs that are meant to explain something to somebody, must contain elements of generalization and therefore they cannot be completely secure.

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It rains almost imperceptibly. In the mountains, it is foggy and clouds hang deeply. Halfway up on the hill a flock of sheep is moving back and forth, without taking a definite direction. The shepherd and his dogs work with the herd. The shepherds work is varied and his experience and competence are quite complex. One has to belong to a family with a long tradition to become a good shepherd or one should be a genius, which in partial ignorance of traditional practices and traditional trade secrets, invents something new, something effective and lasting, thereby changing the tradition and established practice. And one acts mainly on the basis of implicit or tacit knowledge (as opposed to formal, codified or explicit knowledge). It is this a kind of knowledge that is difficult to transfer to another person by means of writing it down or verbalizing it.

In scientific practice, it is the same. It does not hurt to be the child of a scientific researcher or professor. Knowledge comes from knowledge, art comes from art, and science arises from science, education from education. Only as soon as knowledge and craft practices became partially mechanized and automated during the industrial

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revolution and later, professional research and institutionalized learning became common. The well-known chemist and philosopher of science Michael Polanyi (1891–1976) argued that all knowledge is rooted in tacit knowledge in the strong sense of that term.

However, what role then does the teacher play at school? He is not supposed to pass on implicit knowledge and skills to be acquired only in imitation, like the shepherd. In the great education reform of the so-called “modern mathematics” movement, about 50 years ago they wanted to make curricula “teacher-proof”, immune to the interventions and activities of teachers and their possible inadequacies. That did not really work!

We had once called teachers “exemplary intellectuals” (Otte, 1993, cap. VI, VII). We wanted to say that the teacher would influence his students not primarily through pedagogical methods and techniques, but by what he himself is. Not the explicit instructions and the individual teacher's words are decisive, but especially important seems the spirit and the credibility that he radiates in his activities. The teacher acts efficiently primarily by the character of his own intellectual life. The student needs to experience the personalized embodiment of knowledge and of the thoughts that are expressed in it. Theories remain to him mere dry paper if they are not animated by the thought and attitude of a human being. Effective transfer of implicit and explicit knowledge generally presupposes motivation on the side of the learner and this requires an extensive personal contact, regular interaction and trust.

In addition, implicit knowledge includes beliefs, ideals, values, schemata and paradigms, which are deeply ingrained in us and which we often take for granted. They become explicit only after a “scientific revolution”, in the sense of Thomas Kuhn (Kuhn, 1962).

Joshua Reynolds said in his Inaugural Address of the *Royal Academy* in 1769: “Every seminary of learning may be said to be surrounded with an atmosphere of floating knowledge, where every mind may imbibe somewhat congenial to its own original conceptions. Knowledge, thus obtained, has always something more popular and useful than that which is forced upon the mind by private precepts or solitary meditation” Reynolds, *Seven Discourses on Art*, Internet Source).

To understand is to form concepts and concepts arise from concepts and the scientific paradigms, which they express. In empirical contexts, we observe certain regularities, such as distributions of measured data and seek the generating principle

behind the data. This, however, is not possible in a purely inductive manner, just pulling from the data to the law generating them. It requires alongside the data certain general ideas, forms and concepts. William Whewell, the most eminent British philosopher of science of the 19th century, expresses this in a quite charming manner:

“Induction is familiarly spoken of as the process by which we collect a General proposition from a number of particular cases: and it appears frequently imagined that the general proposition results from a mere juxta-position of the cases. ... But if we consider the process more closely ... we shall perceive that this is an inadequate account of the matter. The pearls are there, but they will not hang together till someone provides the string. ... Hence in every inference by Induction, there is some conception *superinduced* upon the facts: and we may henceforth perceive this to be the peculiar import of the term *Induction*” (Whewell, 1847, vol. 2, pp. 46-48).

On the other hand, can ideas “nowise be connected without continuity” (Peirce, CP 143), that is, without acknowledging that ideas are continua of particulars, rather than isolated entities or distinct Platonic ideas or sets. A law of nature becomes visible by a distribution of measured values, after all, although the latter do not per se produce their law-like interpretation. Peirce discusses a famous example, the work of Tycho de Brahe, Copernicus and Kepler criticizing John Stuart Mills empiricist interpretation of this work. Peirce writes:

“Mill denies that there was any reasoning in Kepler’s procedure. He says it is merely description of facts. But so to characterize Kepler’s work is to betray total ignorance of it. What Kepler was given was a large collection of observations of the apparent places of Mars at different times. He also knew that, in a general way, the Ptolemaic theory agrees with the appearances,.... He was furthermore convinced that the hypothesis of Copernicus was to be accepted. But, Kepler did not understand the matter quite as Copernicus did. Kepler looking at the matter dynamically thought it must have something to do with causing the planets to move in their orbits. At each stage of his long investigation, Kepler has a theory, which is approximately true And he proceeds to modify this theory after the most careful and judicious reflection, in such a way as to render it more rational or closer to the observed fact” (Peirce, CP 1.71-1.73).

Therefore, Kepler obviously proceeded simultaneously on two different levels, on the level of data and of ideas. Consider the following example: Catenary and parabola are, especially near the minimum, nearly indistinguishable and Galilei believed, in fact,

that they were equal. Only nearly a hundred years later did Huygens discover that the description of the catenary by an algebraic curve of second degree does not fit completely well. One might think that it is very relative, which point of view one adopts, the geometrical, arguing on base of a continuity principle, or the algebraic, which dominated since Descartes. However, from a physical and practical perspective, it might be reasonable to assume a dynamical viewpoint, as Kepler did, and doing so, by for instance, comparing a hanging chain with a suspension bridge, we realize that the distribution of forces in the two cases are very different and this decides the issue in favor of Huygens.

Even if we do tell the same repeatedly in an always-new manner, it is not necessary the same thing and the outcome is not a chain of mere tautologies. All our reasoning is by signs and the essential thing consists in an interaction of intensional and extensional aspects of our representations. If we abandon this complementarity, situations degenerate.

Since Leibniz (1646-1716), one believed that the true and pure mathematics should be interpreted as an analytical language that only runs the game of the same and the different, and that thus all mathematical knowledge would be a great tautology. This game is a game of forms, rather than referring to some content. Mathematics, it is said, is a part of the logic and logic arises only in this way, that we create a language and a symbolism, which allows us to present the same thing in many different ways, as soon as we have the thing already symbolized in the first place. Formal mathematics and logic speak, according to analytic philosophy, not about objects. “They say nothing about objects, of which we want to speak, but deal only with the way we talk about objects” (Hahn, 1988, p. 150).

“The position of mathematics”, writes Hahn, “has always been of great difficulty to the empirical standpoint, because experience can give us no general knowledge, mathematics, however, seems to be universal knowledge, all empirical knowledge remains somewhat uncertain, in mathematics we do not notice any uncertainty” (Hahn, 1988, p. 55).

In addition, since, with the birth of pure mathematics, mathematical proofs became formal and were employed in the service of clarifying the structure of formal theories, rather than to provide subjective explanations. They have to be based on logically necessary and sufficient premises such that they amount to logical equivalences and thus become tautological.

At the same time, analytical philosophy started its work with the belief that a philosophical explanation of thought can be achieved through a philosophical analysis of language, and that a full explanation can only be achieved in this and no other way. Now it is a strategy of this analysis to explain a concept or a proposition by means of a different one in an informative way. However, if the correct analysis of concepts allows only conceptually true statements, it is to be suspected that these are uninformative, simply because such an analysis is complete only when the involved terms are synonymous, that is, their extensions are identical. If the analyzed concept and its explanation have exactly the same meaning as in extensional mathematics and logic, then we have a “paradox of analysis”, which is sometimes also called the “paradox of proof” (see also, Newen, 2005).

Explanation, in contrast, is always asymmetric, mathematical calculation or logical proof are not. Aristotle has made this very clear already, thereby differentiating between explanation and logical deduction or mathematical calculation. One can calculate the height of the flagpole from the length of its shadow, but the shadow does not produce the flagpole. If one sees a shadow, one looks for a cause and an explanation. If one sees a flagpole there seems to be no question whatsoever. Equally, mathematicians wanted this to happen when seeing a mathematical proof. There should remain no “deep meanings” and everything was to be plain surface and perceivable form.

Another remarkable trait of mathematical language is that it thinks largely for itself such that no questions about meanings arise. “Frequently, students are instructed that they must think about things in order to understand them and to move forward. But in some sense, the greatest progress of human thought have incurred as a result that we have learned to do things without thinking” (Barrow, 1992, p. 3 (our translation)).

This progress led however to a harmful separation of the processes of discovery from the contexts of justification, a separation commonly addressed by the distinction between intuition and logic, as well as between implicit and explicit knowledge. Consider the following paradox: On the hand, a proof can only prove something insofar as the knowledge has a fixed tautological structure and the evidence ultimately consists of the juxtaposition of immediate identities or equalities. Here, the proof, on the other hand, reduces the knowledge to be communicated to the existing knowledge and it is not apparent how this process could possible generate new knowledge. If, therefore, the proof is intended to produce new knowledge - and mathematical knowledge can be

obtained in no other way - then it cannot be a tautological process, which exerts physical or causal force, but must be a semiotic and largely metaphorical process.

Mathematical proof is a communicative device and characterizes no interaction between reactive systems, but between cognitive systems. Mathematical communication must transmit something new to the student and in order to accomplish that, proofs must generalize. Moreover, as generalizations are never secure, but always remain somewhat hypothetical, proofs, which explain, cannot be completely formal proofs.

We teach mathematics at school not only because we believe that it will help to establish and legitimate a discourse, which everybody of good will can accept in good faith, but also because we share Plato's belief that mathematics will lead us to see the truth. Such a belief has been at the bottom of all human aspirations for intelligibility since the times of the Greek. Mathematics could not fruitfully be organized and pursued at school as a primarily professional topic. Mathematical education has, like other subjects, also to contribute to a common search for clarity on fundamental issues.

Here arises the problem of meaning and of intensional semantics, and that has to do with the objectivity of human activity and communication, that is, it does not arise as long as knowledge is considered as a mere mirror image of objective reality. Firstly, one understands the meaning (the sense) as a possible relation and as a transformation and as a translation. In this sense, the meaning of an algebraic equation is in the process of calculation and in the derived and transformed equations. And the meaning of a system of formal axioms is in its logical consequences. On the other hand, however, the meaning may appear as attached to the reference and subordinate to it, as a perspective (among other possible) on some object, or as, as Frege puts it, as "mode of presentation of an object".

Semantics can, consequently be understood in two different ways, namely as the branch of linguistics that deals with the study of meaning and communication, or, secondly, as the study of reference, of the relationships between signs or symbols and what they represent. One could call the first intensional and the second view extensional semantics. Leibniz, Kant and Bolzano were intensionalists in this sense, adhering to the first view of semantics, Frege, Wittgenstein and Cantor endorse the second.

Frege had pointed out that the same individual might have various names, whose meanings are somewhat different. His own classic example was that "Hesperus" is the name of the "Evening Star", while "Phosphorus" is the name of the "Morning Star"; but it turns out that the Evening Star and the Morning Star are the same thing, the planet

Venus. The identity of the object, however, does not make it correct to call Venus in the evening “Phosphorus”. But why not? Would it not be more informative to use a name like “Phosphorus” also referentially? Frege, in order to explain why $A=B$ should be less trivial than $A=A$, introduced the distinction between sense or meaning and reference and assumed that descriptions function like referring expressions.

In Frege’s famous essay on *Sinn und Bedeutung*, the author quotes some examples from elementary geometry. Frege writes:

“Let a, b, c be the lines connecting the vertices of a triangle with the midpoints of the opposite sides. The point of intersection of a and b is then the same as the point of intersection of b and c . So we have different designations for the same point, and these names (‘point of intersection of a and b ’; ‘point of intersection of b and c ’) likewise indicate the mode of presentation, and hence the statement contains actual knowledge”. (Frege, 1975, p. 40).

It is often thought that Frege's puzzle about how ‘ $A = A$ ’ and true ‘ $A = B$ ’ statements can differ semantically, which he uses to motivate the introduction of his notion of sense, cannot be solved without Fregean senses. It can, however, obviously be handled with any notion of sense, since any notion of sense permits us to assign different senses to the symbols A and B . And Frege cannot in fact describe the relation of A and B as parts of $A=B$ without reference to an object: In Frege’s view, meanings or intensions are reduced to mere ways of introducing or presenting an object. There are no meaning relations at stake at all. What one says is that the brightest heavenly body different from the moon which is sometimes seen to precede the rising sun in the east is the same as that heavenly body which is at other times brightly shining in the west after the setting of the sun.

Intensional semantics (Katz, 2004) do not indicate $A = B$, as Frege, in terms of the emphasis on the equal, but are interested in the different and watch out even for the possibly not intended. Equations of the form $A = B = C = \dots$ are initially only promises or hypotheses. And by creating a relationship between two things that is not obvious, a new perspective on reality and an idea are created. The referent, which makes $A = B$ true in cases like the example of economic values or of mathematical entities or of theoretical terms, like energy – of which heat and motion are different manifestations, for instance, - or the electro-magnetic field, or the general triangle, is not necessarily given as such, but is rather a universal idea.

We should remind ourselves that the range of possible applications of a theoretical concept is not a well-defined and predetermined set and, in fact, we cannot know the range of possible applications of a concept in advance, the extension of a concept is not a fixed and predetermined set.

Every act of creative behavior consists, in fact, in seeing an A as a B : $A = B$: a chair as a step-ladder, a hammer as part of a pendulum, a force as a vector, a mechanical operation as a calculation, a movement as a mathematical function, etc. etc.

Jerrold Katz, philosopher and linguist, had criticized Frege's views and has proposed a non-reductive definition of sense or intension. Katz writes:

“The non-reductive definition of sense that I will oppose to the Fregean reductive definition is (D): Sense is that aspect of the grammatical structure of sentences that is responsible for their sense properties and relations (e.g., meaningfulness, meaninglessness, ambiguity, synonymy, redundancy, and antonymy). On (D), senses are still determiners, but what they determine are sense properties and relations, not referential properties and relations. Sense properties and relations, like syntactic properties such as well formedness and phonological properties like rhyme, reflect the grammatical structure within the sentences of a language, in contrast to referential properties and relations, which reflect the connection between language and the world. In taking it to be internal to sentences, (D) makes sense independent of reference, and makes the theory of sense autonomous” (Katz, 2004, p.17).

Languages, says Katz, “are conceived of as game like activities in which the participants use signs in accordance with rules, analogous to the rules of chess and other social practices” (Katz, 1990, p. 3).

Every utterance and every argument is initially nothing more than a move in a language game and has direct significance only in the context of this game and its rules. The question that now arises, is, shall meaning or significance, as Wittgenstein and Rorty following him want, be reduced to the explicit and explicitly sanctioned or shall deviations and hidden ways be admitted, as they are common in poetry, but also in the creative reapplication of mathematical rules and scientific ideas. In other words, shall we adopt Frege's conception of sense or not.

Metaphors, for example, can be seen as the result of the willingness to see something as something else, Napoleon as Roman Emperor, as in the paintings of Jacques-Louis David, for example, or a bathtub or a urinal as a work of art, like Marcel Duchamp A. Danto writes:

“When Napoleon is represented as a Roman emperor, the artist is not just representing Napoleon in an antiquated get-up, the costumes believed to have been worn by the Roman emperors. Rather the artist is anxious to get the viewer to take towards the subject – Napoleon – the attitudes appropriate to the more exalted Roman emperors.... That figure so garbed is a metaphor of dignity authority, grandeur, power and political utterness. Indeed the description or depiction of *A* as *B* has always this metaphoric structure” (Danto, 1981, p. 167).

And Danto continues, saying that “the viewer must perceive the metaphor as an answer to the question of why *that* man has been put by the artist in *those* clothes – a different question entirely from that which asks why Napoleon is dressed that way, the answer to which might not be metaphorical at all ... -the locus of the metaphorical expression is in the representation – in Napoleon – as Roman-emperor – rather than in the reality represented, namely Napoleon wearing those clothes” (Danto, 1981, p. 171).

Napoleon wanted to usurp the authority and dignity of the Roman emperors, which has nothing metaphorical in itself. David's painting of the process belongs to a different context, to the context of art and it is a metaphor, because it provides not only a historical information, but conveys a general sense and an aesthetically mediated access to reality. If we were interested in the information, then a photograph might have been more useful (this, however, had still to be invented). Of course, you can also understand the pictures of David about the coronation of Napoleon literally, as historically informative documents, which only shows that the metaphorical is a question of the representation and perspective, rather than of the objects represented.

It is the transposition into a different medium and a different world, which fascinates. It is also not essential to imitate visual reality, but to analyze it and present it in the light of a new idea. Painting is not simply a matter of copying a flower or a face as exactly as possible. Such a view would never accept Warhol's “Brillo Box” as a work of art. The fascinating thing about Andy Warhol's or Duchamp's art is just the transportation of an ordinary article of daily use into a new context. As A. Danto writes, the question was not, what had made Warhol's “Brillo Box” to become art, “but how, if it was an object of art, objects exactly like it were not” (Danto, 2003, p. XVI).

And, to name one more example, the context of language itself provides Franz Kafka's account of his transformation into an insect with credibility. There seem to exist no limits of imagination. However, somewhere the thing breaks down and then not because of a lack of creativity, but because the meanings become completely detached

from it all. Creativity is contingent, anti-social and revolutionary and that has its dangers and its limits.

This brings us back to our starting point and to the complementarity of intension or meaning and reference. All knowing is by signs. A knowledge, which is to have a formative impact on our minds, must be subjectively meaningful in the first place and that in turn, attracts an interest in a theory of meaning and in the complementarity of the intensional and extensional aspects of sign systems and languages. The above-indicated controversy about intensional and extensional semantics is, as Katz writes, “the central issue of twentieth-century philosophy of language. Properly the issue is not whether sentences of natural languages have Fregean senses, but whether they have senses” (Katz, 2004, p. 7).

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